Notes on Pratt's

"Risk Aversion in the Small and in the Large"

by

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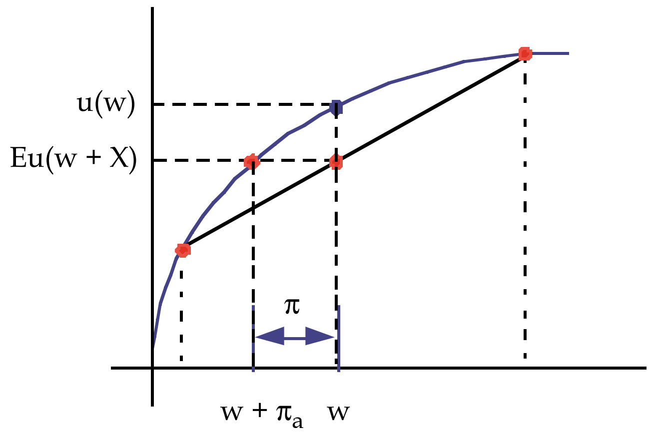
Consider a decision maker with initial assets w and utility function u. The risk premium π is defined as the dollar amount which makes the individual indifferent between the risk X and the non-random amount $EX - \pi = \mu_X - \pi$, i.e., π is implicitly defined by

$$u(w + \mu_X - \pi) = Eu(w + X).$$

The Arrow-Pratt measure of absolute risk aversion is defined by $a = - u''/u'$.[1] The following figure is a graphic interpretation of the risk premium for the case in which the risk X has and expected value of zero. Note that π_a is the cash equivalent of the risk and is defined as $\pi_a = EX - \pi$;[2] the cash equivalent is also the asking price of an agent who owns the risk, i.e., the smallest price for which the agent would be willing to sell the risk. For unfavorable risks it is also possible to define an insurance premium $\pi_i = -\pi_a = \pi - EX$ since receiving the asking price is equivalent to paying the insurance premium. If the expect value of the risk is zero then the insurance premium equals the risk premium

[1] There is also a relative measure of risk aversion. Let r denote that measure and define it as $r(x) = - u''(x) x/u'(x)$.

[2] The cash equivalent is implicitly defined by the condition $u(w + \pi_a) = Eu(w + X)$. Similarly, the insurance premium, introduced subsequently, is implicitly defined by the condition $u(w - \pi_i) = Eu(w + X)$.



One of Pratt's contributions was in showing the relationship between the risk premium π and the risk aversion measure "a." The following is a modified statement and proof of Pratt's theorem.

Pratt Theorem. Let a_i and π_i denote the measure of risk aversion and the risk premium corresponding to the utility function u_i , $i = 1, 2$, respectively. Then the following conditions are equivalent:

(a) $a_1 > a_2$

(b) $\pi_1 > \pi_2$

(c) There exists a transformation T, $T' > 0$, $T'' < 0$, such that $u_1 = T(u_2)$.

Proof. To show that (c) implies (a) note $u_1' = T' u_2'$ and $u_1'' = T' u_2'' + T'' (u_2')^2$. Then

$$a_1 = -\frac{u_1''}{u_1'} = -\frac{T' u_2'' + T'' (u_2'')^2}{T' u_2'} = a_2 - \frac{T''}{T'} u_2' .$$

Hence

$$a_1 - a_2 = - \frac{T''}{T'} u_2' > 0.$$

Next we show that (a) implies (c). Consider the obvious transformation T(z), where

$$T(z) = u_1(u_2^{-1}(z)).$$

Clearly $u_1(z) = T(u_2(z))$ and so it remains to show that T is concave. Note that

$$T'(z) = u_1'(u_2^{-1}(z))\,(u_2^{-1})'(z) = \frac{u_1'(u_2^{-1}(z))}{u_2'(u_2^{-1}(z))}$$

and so

$$T''(z) = \frac{u_2'\left(u_2^{-1}(z)\right) u_1''\left(u_2^{-1}(z)\right)\left(u_2^{-1}(z)\right)' - u_1'\left(u_2^{-1}(z)\right) u_2''\left(u_2^{-1}(z)\right)\left(u_2^{-1}(z)\right)'}{\left(u_2'\left(u_2^{-1}(z)\right)\right)^2}$$

$$= \frac{\dfrac{u_2'\left(u_2^{-1}(z)\right) u_1''\left(u_2^{-1}(z)\right)}{u_2'\left(u_2^{-1}(z)\right)} - \dfrac{u_1'\left(u_2^{-1}(z)\right) u_2''\left(u_2^{-1}(z)\right)}{u_2'\left(u_2^{-1}(z)\right)}}{\left(u_2'\left(u_2^{-1}(z)\right)\right)^2}$$

$$= \frac{u_1'}{\left(u_2'\right)^2}\left\{\frac{u_1''}{u_1'} - \frac{u_2''}{u_2'}\right\}$$

$$= \frac{u_1'}{(u_2')^2} \left\{ a_2 - a_1 \right\}$$

$$< \; 0$$

i.e., this inequality follows by hypothesis.

Next to show that (c) implies (b) note that

$$\pi_i(w, X) = w + \mu_X - u_i^{-1}(Eu_i(w + X)).$$

Therefore

$$\pi_1 - \pi_2 = u_2^{-1}(Eu_2(w + X)) - u_1^{-1}(Eu_1(w + X))$$

$$= \; u_2^{-1}(EY) - u_1^{-1}(ET(Y))$$

where $Y = u_2(w + X)$. Due to Jensen's Inequality we know

$$ET(Y) < T(EY)$$

and so, since u_1^{-1} is an increasing function,

$$u_1^{-1}(ET(Y)) < u_1^{-1}(T(E(Y)).$$

Therefore,

$$\pi_1 - \pi_2 > u_2^{-1}(EY) - u_1^{-1}(T(EY)) = u_2^{-1}(EY) - u_1^{-1}(u_1(u_2^{-1}(EY)) = 0$$

since $u_1^{-1}(u_1(u_2^{-1}(EY) = u_2^{-1}(EY)$.

Finally to show that (b) implies (c) note that $u_2 = T^{-1}(u_1)$ and so $u_2^{-1} = u_1^{-1}(T)$. Then $\pi_1 > \pi_2$ implies

$$u_2^{-1}(EY) > u_1^{-1}(ET(Y))$$

which implies

$$u_1^{-1}(T(E(Y))) > u_1^{-1}(ET(Y)).$$

This condition holds for all random Y if and only if T(E(Y)) > ET(Y).[3] Hence, T is concave.

Problems:

[1] Consider the utility function $u(x) = a_0 + a_1 x + a_2 x^2$ where a_j is a constant, j = 0, 1, 2. Derive and sketch the absolute and relative risk aversion measures. Derive the risk premium, π.

[2] Consider the utility function $u(x) = - e^{-a x}$ where a is a positive constant. Derive and sketch the absolute and relative risk aversion measures. Derive an explicit expression for the risk premium, π.

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[3] If you are wondering whether this statement is true then consider the following argument, suggested to me by Pat Brockett. Let Y be a random variable that takes value x with probability p and value z with probability 1 - p. Note that $ET(Y) < T(EY)$ is equivalent to $p\, T(x) + (1 - p)\, T(z) < T(p\, x + (1 - p)\, z)$. If this inequality holds for all x and z then concavity follows.

[3] Consider the utility function $u(x) = x^p$ where p is a positive constant. Derive and sketch the absolute and relative risk aversion measures. Derive an explicit expression for the risk premium.

[4] Show that maximizing the cash equivalent is equivalent to maximizing expected utility.