Notes on Rothschild-Stiglitz's

"Increasing Risk: I. A Definition"

by

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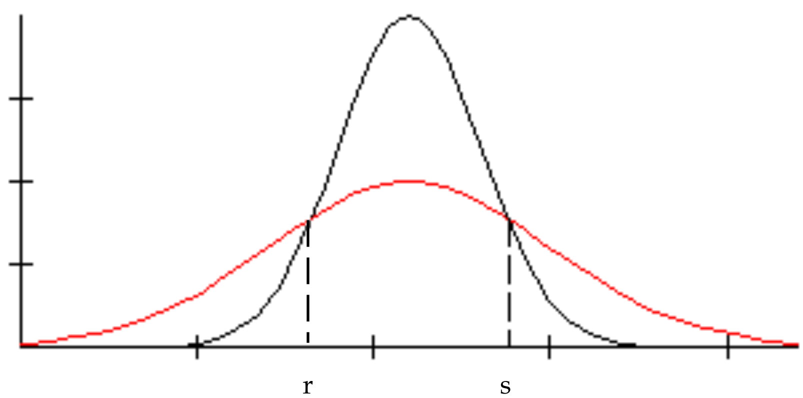
The Rothschild-Stiglitz Theorem

A major result in risk theory, popularized by Rothschild and Stiglitz, concerns the comparison of risks rather than risk aversion measures. As Rothschild and Stiglitz noted, Y may be said to be riskier than X if all risk averse individuals prefer X to Y, or if Y has more weight in the tails of its distribution than X, or if Y equals X plus noise. Rothschild and Stiglitz observed that these three notions of risk are equivalent.

Before proceeding, consider what it means for Y to have more weight in its tails than X. Let F and G be the distribution functions of X and Y, respectively. Similarly, let f and g be the density functions for X and Y, respectively. Consider the densities shown in figure 1. Note that $g(t) \geq f(t)$ for $t \leq r$ and $t \geq s$. This

Figure 1



implies $G(t) > F(t)$ for $t \leq r$ and $G(t) < F(t)$ for $t \geq s$. Also notice that since X and Y are non-negative we have

$$\mu_X = \int_0^\infty \left[1 - F(t)\right] dt^1$$

and

$$\mu_Y = \int_0^\infty \left[1 - G(t)\right] dt$$

Since

$$\int_0^x \left[G(t) - F(t)\right] dt$$

is a decreasing function of x for sufficiently large x and

$$\int_0^\infty \left[G(t) - F(t)\right] dt = 0 \tag{1}$$

it follows that

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[1]To see this, let $F^1 = 1 - F$ and note that

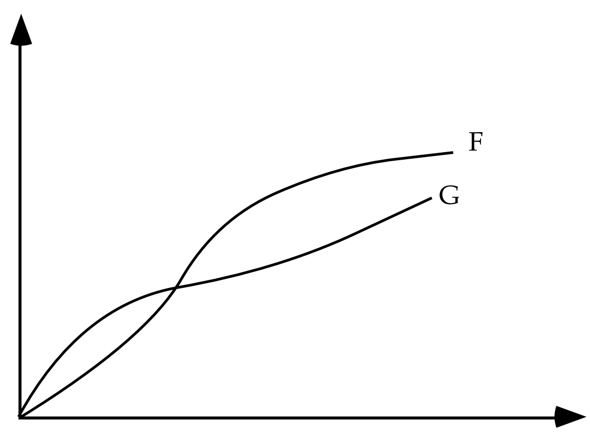
$$\int_0^\infty x\,dF = -\int_0^\infty x\,dF^1 = x\,F^1\Big|_0^\infty + \int_0^\infty F^1\,dx = \int_0^\infty F^1\,dx$$

This obviously holds even if the support of the random variable is compact.

$$\int_0^x \left[G(t) - F(t)\right] dt \geq 0 \tag{2}$$

for all $x \geq 0$. If conditions (1) and (2) are satisfied then $\mu_Y = \mu_X$ and Y has more weight in its tails than X; equivalently, Y is a **mean preserving spread** of X. From figure 1 it is apparent that G and F cross once as shown in figure 2. In this special case we say that Y is a single mean-preserving

Figure 2



spread of X. Note that conditions (1) and (2) apply more generally, i.e., G and F may cross a finite number of times and still satisfy (1) and (2).

The Rothschild-Stiglitz Theorem. Let X and Y be non-negative random variables. The following conditions are equivalent:

(a) $Eu(X) \geq Eu(Y)$, for all $u \in U$, where U is the set of all concave functions;

(b) $\int_0^x \left[G(t) - F(t)\right] dt \geq 0$ for all $x \geq 0$ and $\int_0^\infty \left[G(t) - F(t)\right] dt = 0$;

(c) $Y =^d X + Z$, where $E(Z \mid x) = 0$ for all $x \geq 0$.

Proof. To show that (b) implies (a) note that integrating by parts twice yields

$$Eu(X) = u(0) + u'(\infty)\,\mu_X - \int_0^\infty u''(x)\,F^2(x)\,dx$$

where

$$F^2(x) = \int_0^x (1 - F(t))dt.$$ [2]

[2]Note that for all u

$$Eu(X) = \int_0^\infty u(x)dF(x) = -\int_0^\infty u(x)dF^1(x$$

where $F^1 = 1 - F$. Integration by Parts yields

$$Eu(X) = -u(x)\,F^1(x)\,|_{[0,\infty)} + \int_0^\infty u'(x)\,F^1(x)\,dx = u(0) + \int_0^\infty u'(x)\;F^1(x)\;dx\,.$$

Let $u = u'(x)$ and $dv = F^1(x)\,dx$. It follows that $du = u''(x)\,dx$ and

$$v = \int_0^x F^1(t)\,dt = F^2(x).$$

Then Integration by Parts again yields

$$Eu(X) = u(0) + u'(\infty)\,\mu_X - \int_0^\infty u''(x)\;F^2(x)\;dx\,.$$

Then it follows that

$$\text{Eu(X) - Eu(Y)} = \int_0^{\infty} u''(x)\left[G^2(x) - F^2(x)\right] dx \geq 0$$

since u" ≤ 0 and $G^2 - F^2 \leq 0$, or equivalently,

$$G^2(x) - F^2(x) = \int_0^x \left(G^1(t) - F^1(t)\right) dt = \int_0^x \left(F(t) - G(t)\right) dt \leq 0,$$

where $F^1 = 1 - F$ and $G^1 = 1 - G$.

Next to show that (a) implies (b) note that since u(t) = t and u(t) = - t are concave, it follows that $\mu_Y = \mu_X$. Suppose that $F^2(t^0) < G^2(t^0)$ for some t^0. Then by continuity $F^2(t) - G^2(t) < 0$ for all t in the interval $(t^0 - \delta, t^0 + \delta)$, $\delta > 0$. Take u to be concave with u" = - 1 for t in the interval $(t^0 - \delta, t^0 + \delta)$ and zero elsewhere, i.e., u is linear over $(-\infty, t^0 - \delta]$ and $[t^0 + \delta, \infty)$, and quadratic over $(t^0 - \delta, t^0 + \delta)$. Then by the above calculation

$$\text{Eu(X)} - \text{Eu(Y)} = \int_{t^0-\delta}^{t^0+\delta} \left(F^2(t) - G^2(t)\right) dt \leq 0$$

contradicting the assumption. Hence $F^2 < G^2$ is impossible.

To show that (c) implies (a), note that by Jensen's Inequality

$$Eu(Y) = \int_0^{\infty} \int_{-\infty}^{\infty} u(x+z)\, h(z \mid x)\, f(x)\, dz\, dx$$

$$\leq \int_0^{\infty} u\Big(x + E\{Z \mid x\}\Big)\, f(x)\, dx$$

$$= \int_0^{\infty} u(x)\, f(x)\, dx$$

$$= Eu(X)$$

Finally to show that (a) implies (c) . . . .

One of the most intuitively appealing characterizations of riskiness is the mean preserving spread. Recall that Y is a mean preserving spread of X, and so Y is riskier than X, if $\mu_Y = \mu_X$ and

$$\int_0^x \Big[G(t) - F(t)\Big]\, dt \geq 0$$

for all $x \geq 0$. This is equivalent to $G^2(\infty)$ - $F^2(\infty)$ = μ_Y - μ_X = 0 and $F^2(x) \geq G^2(x)$ for all x. There are two ways in which to restate the condition $F^2(x) \geq G^2(x)$. For example $F^2(t) \geq G^2(t)$ is equivalent to $E\min\{Y, t\} \geq E\min\{X, t\}$. Hence if $\mu_Y = \mu_X$ and $E\min\{Y, t\} \geq E\min\{X, t\}$ then Y is riskier than X. The two equivalent characterizations are provided in the following corollary to the Rothschild-Stiglitz theorem.

Corollary. Given non-negative absolutely continuos random variables X and Y, the following are equivalent:

(a) Y is riskier than X in the Rothscild-Stiglitz sense;

(b) $E\min\{t, X\} \geq E\min\{t, Y\}$ for all $t \geq 0$;

(c) $E\max\{t, Y\} \geq E\max\{t, X\}$ for all $t \geq 0$;

(d) $E\max\{0, Y - t\} \geq E\max\{0, X - t\}$ for all $t \geq 0$;

(e) $E\max\{0, t - Y\} \geq E\max\{0, t - X\}$ for all $t \geq 0$.

Proof. To demonstrate the equivalence of (a) and (b) note that

$$F^2(t) = \int_0^t F^1(x)\,dx = F^1(x)\,x\Big|_0^t + \int_0^t x\,dF(x) = E\min\{t, X\}.$$

Thus Y riskier than X is equivalent to X preferred to Y for all concave u is equivalent to $F^2(t) \geq G^2(t)$ for all t which is equivalent to $E\min\{t, X\} \geq E\min\{t, Y\}$ for all t.

Next, to demonstrate the equivalence of (b) and (c) simply note that

$$\min\{t, x\} + \max\{t, x\} = t + x$$

and so

$$E\min\{t, X\} + E\max\{t, X\} = t + \mu_X.$$

It follows that $E\max\{t, Y\} - E\max\{t, X\} = E\min\{t, X\} - E\min\{t, Y\} \geq 0$.

Next, observe that $\max\{t, X\} - t = \max\{0, X - t\}$. Hence, $\max\{t, Y\} - \max\{t, X\} = \max\{0, Y - t\} - \max\{0, X - t\}$ and (d) follows immediately from (c).

Finally, observe that $\max\{t, X\} - X = \max\{0, t - X\}$. By (c)

$$E\max\{t, Y\} - EY \geq E\max\{t, X\} - EX$$

or equivalently $E\max\{0, t - Y\} \geq E\max\{0, t - X\}$. Q.E.D.

Reference

Rothschild, M. and J. E. Stiglitz (1970). "Increasing Risk: I. A Definition." Journal of Economic Theory **2**: 225-43.