THE EXPECTED UTILITY THEOREM

by

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To characterize the uncertain environment, let the set Ω encompass all possible states of the world. Let ω be an element of the set Ω and let B be a sigma algebra of Ω. A sigma algebra B is a collection of subsets of Ω which satisfies the following properties:

(i) $\Omega \quad B$;

(ii) If $E \quad B$ then $E^c \quad B$;

(iii) If $E_1, E_2, \ldots, E_N \quad B$ then $\;_i E_i \quad B$.

E is a subset of the set of states of nature which we call an event. Then B may be referred to as the event space. Next, let A denote the set of actions available to the decision maker and let a A which denotes a particular action. Let X be a function which maps $A \times \Omega$ into C where C is the set of consequences. Let the event E be defined as $E = \{\omega \quad \Omega \mid X(a, \omega) = c\}$. Now the action "a" specifies the probability of each consequence via the function X and the relation

$$P_a(c) = P\{E\} = P\{\omega \quad \Omega \mid X(a, \omega) = c\}$$

for all c C. $P_a(c)$ is the probability that the consequence "c" will occur when the decision "a" is selected. P_a may be viewed as a "probabilistic consequence" corresponding to "a", since P_a specifies the probabilistic mechanism by which consequences are generated once "a" is selected.[1]

The basic idea of expected utility theory is the following: We already have a complete ordering of the consequences. If we had a complete ordering of probability measures on the set

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[1] This development of the expected utility theorem is based on Bertsekas' *Dynamic Programming and Stochastic Control*, Academic Press, 1976.

of consequences then we could in turn obtain a complete ranking of all actions in A. This is true simply because we could rank two actions a and a' according to the relative order of their corresponding probability measures P_a and $P_{a'}$, i.e., a' is preferred to a if and only if $P_{a'}$ is preferred to P_a. The fundamental premise of expected utility theory is that the decision maker has a complete ordering of all probability measures on the set of consequences, i.e., the decision maker is in a position to express his preference between any two probability distributions on the set of consequences. This settles the question of ranking decisions in view of the preceding relation. Further, if there exists a real valued function G by means of which preferences on the set of distributions can be expressed, i.e., $P_{a'}$ is preferred to P_a if and only if $G(P_{a'}) > G(P_a)$, then decisions can be ranked by means of a real valued function H, i.e., a' is preferred to a if and only if $H(a') > H(a)$, where $H(a) = G(P_a)$ for all a   A and H is the expected utility function.

The interesting aspect of this analysis is that the ordering of decisions can be expressed not only by a function G but also by means of an essentially unique function called the utility function. This function u maps the space of consequences into the set of real numbers and satisfies the relation: a_2 is preferred to a_1 if and only if P_2 is preferred to P_1, i.e., denoted by $P_2 > P_1$, if and only if

$$E\{u(X) \mid a_2\} > E\{u(X) \mid a_1\}.$$

The problem of selecting an action is thus reduced to the problem of maximizing expected utility over the space A.

We will demonstrate the existence of the utility function for the case in which the set of consequences is finite. Let $C = \{c_1, c_2, \ldots, c_N\}$. Let P denote the set of probability distributions $P = (p_1, p_2, \ldots, p_N)$ on C where p_j is the probability of consequence c_j, $j = 1, 2, \ldots, N$. For any P_1, P_2 P, $P_1 = (p_1^1, p_2^1, \ldots, p_N^1)$, $P_2 = (p_1^2, p_2^2, \ldots, p_N^2)$, and any α in [0, 1], we let

$$\alpha\ P_1 + (1 - \alpha)\ P_2 =$$

$$(\alpha\ p_1^1 + (1 - \alpha)\ p_1^2, \ldots, \alpha\ p_N^2 + (1 - \alpha)\ p_N^2).$$

Now, suppose the following axioms hold:

AXIOM 1. There exists a complete and transitive relation "≥" on P. For any P_1, P_2 P we write $P_1 \sim P_2$ if $P_2 \geq P_1$ and $P_1 \geq P_2$, and we write $P_2 > P_1$ if $P_2 \geq P_1$ but not $P_2 \sim P_1$.

AXIOM 2. If $P_1 \sim P_2$, then for all α [0, 1] and all P P, $\alpha\ P_1 + (1 - \alpha)\ P \sim \alpha\ P_2 + (1 - \alpha)\ P$.

AXIOM 3. If $P_2 > P_1$, then for all α (0, 1] and all P P, $\alpha\ P_2 + (1 - \alpha)\ P > \alpha\ P_1 + (1 - \alpha)\ P$.

AXIOM 4. If $P_3 > P_2 > P_1$, there exists an α (0, 1) such that $P_2 \sim \alpha\ P_1 + (1 - \alpha)\ P_3$.

Axiom one simply ensures that the agent is able to make pairwise comparisons and that the preference relation is consistent, or equivalently, transitive, i.e., if $P_3 \geq P_2$ and $P_2 \geq P_1$ then $P_3 \geq P_1$. These assumptions are used in all modern developments of utility theory. Axioms two and three are quite similar. Axiom three says that if P_2 is preferred to P_1 then even a chance α of P_2 is preferrable. Axiom four says that however desirable P_3 may be, its influence can be made as weak as desired by giving it a sufficiently small chance in the lottery. This axiom is simply a plausible continuity assumption.

THE EXPECTED UTILITY THEOREM

The following theorem is the foundation of expected utility theory. It shows that if axioms one through four hold then the preference relation may be represented using an essentially unique utility function.

EXPECTED UTILITY THEOREM. Given axioms one through four, there exists a real valued function u: $C \to R$ called the utility function, such that for all P_1, P_2 P, $P_2 \geq P_1$ if and only if

$$\sum p_i^2 \, u(c_i) \geq \sum p_i^1 \, u(c_i). \qquad (*)$$

Furthermore, u is unique up to a positive linear affine transformation.

Proof. First, we claim that the following statement is true: If $P_3 > P_1$ and P_2 is such that $P_3 \geq P_2 \geq P_1$, then there exists a unique scalar α in the closed interval [0, 1] such that

$$\alpha \, P_1 + (1 - \alpha) \, P_3 \sim P_2. \qquad (1)$$

Furthermore, if P_2' is such that $P_3 \geq P_2' \geq P_2 \geq P_1$ and α' corresponds to P_2' as in (1), then $\alpha \geq \alpha'$.

Note that if $P_3 \geq P_2 \sim P_1$, then $\alpha = 1$ is the unique scalar satisfying (1) since if for some $0 \leq \alpha < 1$ we had

$$\alpha \, P_1 + (1 - \alpha) \, P_3 \sim P_2 \sim \alpha \, P_1 + (1 - \alpha) \, P_2,$$

then axiom three would be contradicted.[2] Similarly if $P_3 \sim P_2 > P_1$, then $\alpha = 0$ is the unique scalar satisfying (1). Assume now that $P_3 > P_2 > P_1$. Then by axiom four there exists an α_1 (0, 1) satisfying (1). Assume that α_1 is not unique and there exists another scalar α_2 (0, 1) such that (1) is satisfied, i.e.,

$$\alpha_1 P_1 + (1 - \alpha_1) P_3 \sim P_2 \sim \alpha_2 P_1 + (1 - \alpha_2) P_3. \qquad (2)$$

Assume that $0 < \alpha_1 < \alpha_2 < 1$. Then we have

$$P_3 = \frac{\alpha_2 - \alpha_1}{1 - \alpha_1} P_3 + \frac{1 - \alpha_2}{1 - \alpha_1} P_3, \qquad (3)$$

and

$$\alpha_2 P_1 + (1 - \alpha_2) P_3 = \qquad (4)$$

$$\alpha_1 P_1 + (1 - \alpha_1)\left\{ \frac{\alpha_2 - \alpha_1}{1 - \alpha_1} P_1 + \frac{1 - \alpha_2}{1 - \alpha_1} P_3 \right\}$$

Since $P_3 > P_1$ by axiom three and by equation (3)

$$P_3 = \frac{\alpha_2 - \alpha_1}{1 - \alpha_1} P_3 + \frac{1 - \alpha_2}{1 - \alpha_1} P_3 \qquad (5)$$

$$> \frac{(\alpha_2 - \alpha_1)}{(1 - \alpha_1)} P_1 + \frac{(1 - \alpha_2)}{(1 - \alpha_1)} P_3.$$[3]

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[2]Recall that axiom three says that if $P_3 \geq P_2$ then for all α (0, 1] and all P P, $(1 - \alpha) P_3 + \alpha P \geq (1 - \alpha) P_2 + \alpha P$; of course if $\alpha = 0$ then we simply contradict $P_3 > P_2$.

[3]Equivalently, by axiom three $P_3 \geq P_1$ β (0, 1] $\beta P_3 + (1 - \beta) P > \beta P_1 + (1 - \beta) P$. Now, simply let $\beta \equiv (\alpha_2 - \alpha_1)/(1 - \alpha_1)$ and let $P \equiv P_3$.

Using this result and again using axiom three and equation (4) we have

$$\alpha_2 P_1 + (1 - \alpha_2) P_3 =$$

$$\alpha_1 P_1 + (1 - \alpha_1) \left\{ \frac{(\alpha_2 - \alpha_1)}{(1 - \alpha_1)} P_1 + \frac{(1 - \alpha_2)}{(1 - \alpha_1)} P_3 \right\} <$$

$$\alpha_1 P_1 + (1 - \alpha_1) P_3.$$

The equality follows by (4) and the preference relation "<" follows by axiom three and equation (5). Since this contradicts (2), it follows that the scalar must be unique.

Next, to show that if $P_3 \geq P_2' \geq P_2 \geq P_1$ then $\alpha \geq \alpha'$ assume the contrary, i.e., $\alpha < \alpha'$. Then, by axioms three and four,

$$P_2' \sim \alpha' P_1 + (1 - \alpha') P_3 =$$

$$(1 - \alpha + \alpha') \left\{ \frac{\alpha}{(1 - \alpha + \alpha')} P_1 + \frac{(1 - \alpha')}{(1 - \alpha + \alpha')} P_3 \right\} + (\alpha' - \alpha) P_1 <$$

$$(1 - \alpha + \alpha') \left\{ \frac{\alpha}{(1 - \alpha + \alpha')} P_1 + \frac{(1 - \alpha')}{(1 - \alpha + \alpha')} P_3 \right\} + (\alpha' - \alpha) P_3 =$$

$$\alpha P_1 + (1 - \alpha) P_3 \sim P_2.$$

Hence $P_2 > P_2'$ which contradicts the assumption $P_2' \geq P_2$. It follows that $\alpha \geq \alpha'$ and the statement is proved.

Now consider the probability distributions $P_1 = (1, 0, \ldots, 0)$, $P_2 = (0, 1, 0, \ldots, 0)$, . . . , and $P_N = (0, 0, \ldots, 0, 1)$. Assume, without loss of generality, that $P_N \geq \ldots \geq P_2 \geq P_1$ and assume further that $P_N > P_1$. If $P_1 \sim P_2 \sim \cdots \sim P_N$, then the proof of the proposition is trivial. Let A_1, A_N be any scalars with $A_1 < A_N$ and define $u(c_1) = A_1$ and $u(c_N) = A_N$. Let α_i, $i = 1, \ldots, N$, be the unique scalar in [0, 1] such that

$$\alpha_i P_1 + (1 - \alpha_i) P_N \sim P_i \tag{6}$$

for $i = 1, \ldots, N$, and define

$$u(c_i) = A_i = \alpha_i A_1 + (1 - \alpha_i) A_N \tag{7}$$

for $i = 1, \ldots, N$. We will show that the function $u: C \rightarrow R$ defined above has the desired property (*). For any probability distribution $P = (p_1, \ldots, p_N)$ it is easily shown that $P_N \geq P \geq P_1$ and so we can define $\alpha(P)$ to be the unique scalar in [0, 1] such that

$$\alpha(P) P_1 + (1 - \alpha(P)) P_N \sim P. \tag{8}$$

From the statement we have proven, we obtain for all P and P', $P' \geq P$ if and only if $\alpha(P) \geq \alpha(P')$. From (6) we have

$$P = \sum p_i P_i$$

$$\sim \sum p_i \left[\alpha_i P_1 + (1 - \alpha_i) P_N\right]$$

$$\sim \sum p_i \alpha_i P_1 + \left(1 - \sum p_i \alpha_i\right) P_N. \tag{9}$$

Comparing (8) and (9) we see that

$$\alpha(P) = \sum p_i \alpha_i.^4$$

Therefore we see that P' ≥ P if and only if

$$\sum p_i' \alpha_i \leq \sum p_i \alpha_i.^5 \tag{10}$$

From (7) we have $\alpha_i = (A_N - A_i)/(A_N - A_1)$, and substituting in (10) we obtain P' ≥ P if and only if

$$\sum p_i' \frac{A_N - A_i}{A_N - A_1} \leq \sum p_i \frac{A_N - A_i}{A_N - A_1}$$

or equivalently,

$$\sum p_i' A_i \geq \sum p_i A_i \Leftrightarrow \sum p_i' u(c_i) \geq \sum p_i u(c_i),$$

which is equivalent to (∗), which was to be proved.

It remains to show that u is unique up to a positive linear affine transformation. If u* is another utility function satisfying (∗) then by denoting $u^*(c_i) = A_i^*$, i = 1,. . ., N, by (∗) and (7) we have

[4]Since $P_1 = (1, 0, 0, \ldots, 0)^T$, $\alpha_i P_1 = \alpha_i$ and the result follows easily from (8) and (9).
[5]Recall that $P' \geq P \Leftrightarrow \alpha' \leq \alpha$. Since $\alpha' = \Sigma p_i' \alpha_i$, the result follows.

$$u^*(c_i) = \alpha_i\, u^*(c_i) + (1 - \alpha_i)\, u^*(c_i).$$

This implies

$$\alpha_i = \frac{A_N^* - A_i^*}{A_N^* - A_1^*} = \frac{A_N - A_i}{A_N - A_1}$$

from which

$$A_i^* = A_N^* - (A_N^* - A_1^*)\frac{A_N - A_i}{A_N - A_1} =$$

$$A_N^* - \frac{(A_N^* - A_1^*)\, A_N}{A_N - A_1} + \frac{A_N^* - A_1^*}{A_N - A_1}\, A_i.$$

Hence u* is a linear affine transformation of u.[6] Q.E.D.

Utility is an ordinal concept in economic theory, i.e., any increasing transformation of the utility function yields the same preferences. Cardinal utility is, in fact, usually associated with pejorative statements. Von Neumann and Morgenstern, however, provide a different perspective.

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[6]This is equivalent to u* = h + k u, where

$$h \equiv A_N^* - \frac{(A_N^* - A_1^*)\, A_N}{A_N - A_1}$$

and

$$k \equiv \frac{A_N^* - A_1^*}{A_N - A_1}.$$

"Given a physical quantity, the system of transformations up to which it is described by numbers may vary in time, i.e. with the stage of development of the subject. Thus temperature was originally a number only up to any monotone transformation. With the development of thermometry-particularly of the concordant ideal gas thermometry-the transformations were restricted to the linear ones, i.e. only the absolute zero and the absolute unit were missing. Subsequent developments of thermodynamics even fixed the absolute zero so that the transformation system in thermodynamics consists only of the multiplication by constants. Examples could be multiplied but there seems to be no need to go into this subject further.

For utility the situation seems to be of a similar nature. One may take the attitude that the only 'natural' datum in this domain is the relation 'greater,' i.e. the concept of preference. In this case utilities are numerical up to a monotone transformation. This is, indeed, the generally accepted standpoint in economic literature, best expressed in the technique of indiffernece curves."[7]

[7]See von Neumann and Morgenstern's *Theory of Games and Economic Behavior*, John Wiley & Sons, 1944, p. 23.