

Notes on Tobin's  
Liquidity Preference as Behavior toward Risk

By

Richard MacMinn

Revised June 1987  
Revised subsequently

Tobin (Tobin 1958) considers a portfolio model in which there is one safe and one risky asset. The net rate of return is  $X_0$  on the safe asset and  $X_1$  on the risky asset.

A portfolio consists of a dollar amount  $\alpha_0$  invested in the safe asset and an amount  $\alpha_1$  invested in the risky asset where  $\alpha_0 + \alpha_1 = w$  and  $w$  is the investor's initial wealth. Let  $Y$  be a random variable denoting the return on the portfolio. Then

$$\begin{aligned} Y &= \alpha_0(1 + x_0) + \alpha_1(1 + x_1) \\ &= w(1 + x_0) + \alpha_1(x_1 - x_0) \end{aligned} \tag{1}$$

and

$$\begin{aligned} EY &\equiv \mu_Y \\ &= w(1 + x_0) + \alpha_1(\mu_1 - x_0) \\ \text{Var } Y &\equiv \sigma_Y^2 \\ &= \alpha_1^2 \text{Var } x_1 \\ \sigma_Y &= \alpha_1 \sigma_1 \end{aligned} \tag{2}$$

The amount  $\alpha_1$  determines both  $\mu_Y$  and  $\sigma_Y$ . The terms on which an investor can obtain a greater expected return at the expense of more risk is

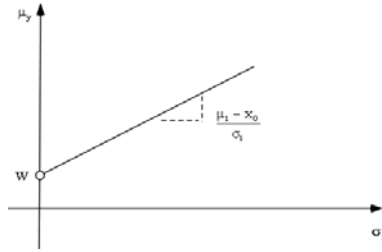
$$\mu_Y = w(1 + x_0) + \frac{\mu_1 - x_0}{\sigma_1} \sigma_Y \tag{3}$$

where  $0 \leq \sigma_Y \leq w\sigma_1$ . (3) is the investor's trading line. The investor holds the amount  $\alpha_1 = \sigma_Y / \sigma_1$  of the risky asset.

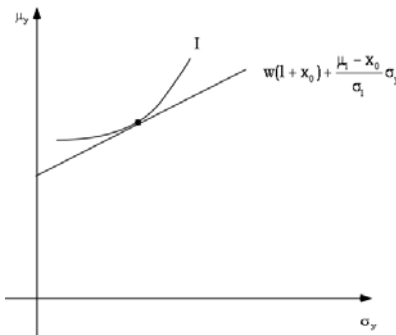
Let  $\alpha_1 = \lambda_1 w$ . Then

$$\begin{aligned} \mu_Y - w &= wx_0 + \frac{\mu_1 - x_0}{\sigma_1} \lambda_1 w \sigma_1 \\ \frac{\mu_Y - w}{w} &= x_0 + \frac{\mu_1 - x_0}{\sigma_1} \lambda_1 \sigma_1 \end{aligned} \tag{4}$$

This is called the capital market line where  $\lambda_1 \sigma_1$  is the standard deviation per unit of wealth held in the risky asset and  $\frac{\mu_1 - x_0}{\sigma_1}$  is sometimes called market price of risk.



The investor is assumed to have preferences which rank all  $(\sigma_Y, \mu_Y)$  pairs and these preferences can be represented by indifference curves. Then the individual maximizes expected utility subject to selecting a pair  $(\sigma_Y, \mu_Y)$  on the trading line.



There are two rationales for supposing that

$$E\mu(Y) = f(\mu_Y, \sigma_Y) \tag{5}$$

The first is that  $Y \sim N(\mu_Y, \sigma_Y)$ . Let  $g(y; \mu_Y, \sigma_Y)$  be the density function of  $Y$ . Then

$$\begin{aligned} E\mu(Y) &= \int_{-\infty}^{\infty} \mu(y)g(y; \mu_Y, \sigma_Y) dy \\ &= f(\mu_Y, \sigma_Y) \end{aligned} \tag{6}$$

and the shape of the indifference curve can be inferred from the shape of  $\mu$ . To demonstrate this we perform the following transformation.

Let

$$\begin{aligned} Y &= \Phi(z) \equiv \mu_Y + \sigma_Y z \\ z &= \phi(Y) \equiv \frac{Y - \mu_Y}{\sigma_Y} \end{aligned} \tag{7}$$

i.e.  $\phi = \Phi^{-1}$ . Then

$$\begin{aligned}
E\mu(Y) &= E\mu(\Phi(z)) \\
&= \int_{-\infty}^{\infty} \mu(\Phi(z)) h(z; 0, 1) dz \\
&= f(\mu_Y, \sigma_Y)
\end{aligned} \tag{8}$$

where  $z \sim N(0,1)$  and  $h$  is the standard normal density. Let  $D_1f$  and  $D_2f$  denote the partial derivatives of  $f$  with respect to  $\mu_Y$  and  $\sigma_Y$  respectively. Then

$$\begin{aligned}
D_1f &= \int_{-\infty}^{\infty} \mu' h \\
D_2f &= \int_{-\infty}^{\infty} \mu' zh
\end{aligned} \tag{9}$$

Since  $f$  is constant on the indifference curve

$$0 = df = D_1f d\mu_Y + D_2f d\sigma_Y \tag{10}$$

or equivalently

$$\begin{aligned}
\frac{d\mu_Y}{d\sigma_Y} &= -\frac{D_2f}{D_1f} \\
&= -\frac{\int_{-\infty}^{\infty} zu'(\Phi(z))h(z)}{\int_{-\infty}^{\infty} u'(\Phi(z))h(z)}
\end{aligned} \tag{11}$$

Claim: if  $u' > 0$  and  $u'' < 0$  then  $d\mu_Y/d\sigma_Y > 0$ .

Proof. Since the denominator is clearly positive, it suffices to show that  $u'' < 0$  implies

$$N \equiv \int_{-\infty}^{\infty} zu'(\Phi(z))h(z) < 0 \tag{12}$$

Note that

$$N = \lim_{a \rightarrow \infty} \left\{ \int_{-a}^0 zu' h + \int_0^a zu' h \right\} \tag{13}$$

where  $h$  is symmetric, i.e.  $h(t) = h(-t)$ . Let  $z = -t$ . Then

$$\begin{aligned}
\int_{-a}^0 zu'(\Phi(z))h(z) dz &= \\
\int_a^0 -tu'(\Phi(-t))h(-t)(-dt) &
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
N &= \lim_{a \rightarrow \infty} \left\{ -\int_0^a tu'(\Phi(-t))h(-t) dt + \int_0^a tu'(\Phi(t))h(t) dt \right\} \\
&= \lim_{a \rightarrow \infty} \left\{ \int_0^a t [u'(\Phi(t)) - u'(\Phi(-t))] h(t) \right\} < 0
\end{aligned} \tag{15}$$

Since  $u'(\Phi(-t)) > u'(\Phi(t))$ .

## Comparative Statistics

Suppose that  $u$  is an increasing concave function and that  $X_1 \sim N(\mu_1, \sigma_1^2)$ . Now suppose  $\mu_1$  changes. Then we want to determine what effect this will have on the investor's optimal choice of  $\alpha_1$ .

Recall that

$$\begin{aligned}\Phi(z) &= \mu_Y + \sigma_Y z \\ &= w(1 + x_0) + \alpha_1(\mu_1 - x_0) + \alpha_1 \sigma_1 z\end{aligned}\quad (16)$$

and the investor's problem is to select  $\alpha_1$  to maximize  $\int_{\square} u(\Phi(z))h(z)$ . The first order condition is

$$\int_{\square} u'(\Phi(z))(\mu_1 - x_0 + \sigma_1 z)h(z) = 0 \quad (17)$$

or equivalently

$$\frac{\mu_1 - x_0}{\sigma_1} = - \frac{\int_{\square} u' z h}{\int_{\square} u' h} \quad (18)$$

Now let the function  $F : D \rightarrow \square, D \subset \square^2$  be defined by

$$F(\mu_1, \alpha_1) = \int_{\square} u'[\mu_1 - x_0 + \sigma_1 z]h \quad (19)$$

Since  $D_2 F < 0$ , it follows by the Implicit Function Theorem that there exists a differentiable function  $f : \square \rightarrow \square$  such that  $F(\mu_1, f(\mu_1)) = 0$  and  $f' = -D_1 F / D_2 F$ .

Note that  $f' \geq 0$  as  $D_1 F \geq 0$ .

$$\begin{aligned}D_1 F &= \int_{\square} \{u' + u'' \alpha_1 [\mu_1 - x_0 + \sigma_1 z]\}h \\ &= \int_{\square} u' h + \alpha_1 \int_{\square} u'' [\mu_1 - x_0 + \sigma_1 z]h > 0\end{aligned}\quad (20)$$

if the second integral on the RHS is non-negative. To determine the sign let  $a \equiv -u''/u'$  denote the measure of absolute risk aversion. Assume that  $a' \leq 0$  and let  $z_0$  be implicitly defined by  $\mu_1 - x_0 + \sigma_1 z_0 = 0$  (i.e.  $z_0 = -\frac{\mu_1 - x_0}{\sigma_1}$ ).

Then

$$-\frac{u''(\Phi(z_0))}{u'(\Phi(z_0))} \leq a(\Phi(z_0)), z \geq z_0 \quad (21)$$

or

$$u''(\Phi(z_0)) \geq -a(\Phi(z_0))u'(\Phi(z_0)) \quad (22)$$

which in turn yields

$$u''(\Phi(z_0))(\mu_1 - x_0 + \sigma_1 z) \geq -a(\Phi(z_0))u'(\Phi(z_0))(\mu_1 - x_0 + \sigma_1 z) \quad (23)$$

for  $z \geq z_0$ . A similar argument shows that (23) holds for  $z < z_0$ . Hence, given  $a' \leq 0$ ,

$$\int_{\square} u''(\Phi(z))(\mu_1 - x_0 + \sigma_1 z) h \geq -a(\Phi(z_0)) \int_{\square} u'(\Phi(z))(\mu_1 - x_0 + \sigma_1 z) h \quad (24)$$

but the RHS is zero at the optimal  $\alpha_1$ . Hence,  $D_1 F \geq 0$  or equivalently  $f'(\mu_1) > 0$ . Note that in Tobin's model (i.e. where  $x_0 \equiv 0$  and so the safe asset is money)  $f'(\mu_1) > 0$  also implies that the demand for money is a decreasing function of the expected return on the risky asset, which he calls the rate of interest.

Next consider a change in the riskness of asset one. Let  $G: D \rightarrow \square, D \subset \square^2$ , be defined as

$$G(\sigma_1, \alpha_1) = \int_{\square} u'(\Phi(z))(\mu_1 - x_0 + \sigma_1 z) h \quad (25)$$

Then as above there exist a function  $g$  such that  $G(\sigma_1, g(\sigma_1)) = 0$  and  $g' = -\frac{D_1 G}{D_2 G} \geq 0$  as  $D_1 G \geq 0$ .

$$\begin{aligned} D_1 G &= \int_{\square} [u'z + \alpha_1 u''z(\mu_1 - x_0 + \sigma_1 z)] h \\ &= \int_{\square} u'zh + \alpha_1 \int_{\square} u''z(\mu_1 - x_0 + \sigma_1 z) h \end{aligned} \quad (26)$$

Recall that the first integral on the RHS is negative and so  $D_1 G < 0$  if the second integral on the RHS is non-positive. Since

$$\begin{aligned} \int_{\square} u''z(\mu_1 - x_0 + \sigma_1 z) h &= \frac{1}{\sigma_1} \int_{\square} u''\sigma_1 z(\mu_1 - x_0 + \sigma_1 z) h = \\ \frac{1}{\sigma_1} \int_{\square} u''[(\mu_1 - x_0) + \sigma_1 z - (\mu_1 - x_0)](\mu_1 - x_0 + \sigma_1 z) h &= \\ \frac{1}{\sigma_1} \left\{ \int_{\square} u''(\mu_1 - x_0 + \sigma_1 z)^2 h - (\mu_1 - x_0) \int_{\square} u''(\mu_1 - x_0 + \sigma_1 z) h \right\} \end{aligned} \quad (27)$$

It follows that  $a' \leq 0$  and  $\mu_1 - x_0 \geq 0$  suffice to show  $D_1 G < 0$ .

### Model with Many Risky Assets

Suppose there are  $N$  risky assets. Let  $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$  where  $\alpha_j, j=1, \dots, N$  is the dollar amount held of risky asset  $j$  and  $\alpha_0$  is defined as before. Let  $x_j$  denote the random rate of return on asset  $j$ . Then

$$\begin{aligned} Y &= \alpha_0 (1 + x_0) + \sum_{j=1}^N \alpha_j (1 + x_j) \\ &= w(1 + x_0) + \sum_{j=1}^N \alpha_j (x_j - x_0) \end{aligned} \quad (28)$$

since

$$\alpha_0 = w - \sum_{j=1}^N \alpha_j$$

Let

$$\sigma_{ij} = E(x_i x_j) - E x_i E x_j$$

Then

$$\mu_Y = w(1 + x_0) + \sum_{j=1}^N \alpha_j (\mu_j - x_0) \quad (29)$$

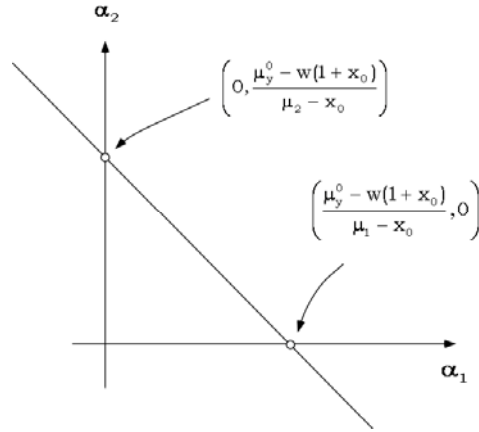
and

$$\begin{aligned} \sigma_Y^2 &= \text{Var} \left( \sum_{j=1}^N \alpha_j x_j \right) \\ &= \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \sigma_{ij} \end{aligned} \quad (30)$$

In the special case  $N=2$

$$\begin{aligned} \text{Var}(\alpha_1 x_1 + \alpha_2 x_2) &= \\ E(\alpha_1 x_1 + \alpha_2 x_2 - (\alpha_1 \mu_1 + \alpha_2 \mu_2))^2 &= \\ E(\alpha_1 (x_1 - \mu_1) + \alpha_2 (x_2 - \mu_2))^2 &= \\ \alpha_1^2 E(x_1 - \mu_1)^2 + 2\alpha_1 \alpha_2 E[(x_1 - \mu_1)(x_2 - \mu_2)] + \alpha_2^2 E(x_2 - \mu_2)^2 &= \alpha_1^2 \sigma_{11} + 2\alpha_1 \alpha_2 \sigma_{12} + \alpha_2^2 \sigma_{22} \end{aligned} \quad (31)$$

The set of points  $\alpha \in \mathbb{R}^N$  for which  $\mu_Y$  is constant is a hyperplane. In the case  $N=2$  the  $\{\alpha \in \mathbb{R}^2 \mid w(1 + x_0) + \alpha(\mu - ex_0) = \mu_Y^0\}$  where  $e = (1, \dots, 1) \in \mathbb{R}^N$  and  $\mu = (\mu_1, \dots, \mu_N)$  is shown



Let  $\Sigma$  denote the variance-covariance matrix. Then  $\sigma_Y^2 = \alpha^T \Sigma \alpha$ . For  $N=2$  the  $\{\alpha \in \mathbb{R}^2 \mid w(1 + x_0) + \alpha(\mu - ex_0) = \mu_Y^0\}$  where  $e = (1, \dots, 1) \in \mathbb{R}^N$  and  $\mu = (\mu_1, \dots, \mu_N)$  is shown

Let  $\Sigma$  denote the variance-covariance matrix. Then  $\sigma_Y^2 = \alpha^T \Sigma \alpha$ . For  $N=2$

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \quad (32)$$

Note  $\Sigma$  is a real symmetric matrix and so there exists an orthogonal matrix  $U$  such that  $U^T \Sigma U$  is a diagonal matrix whose diagonal elements are the characteristic roots of  $\Sigma$ . Since  $\Sigma$  is a real symmetric matrix, all its characteristic roots are real. Let  $U^T \Sigma U = D(\lambda)$  where  $\lambda \in \mathbb{R}^n$  is the vector of characteristic roots, e. g.

$$D(\lambda) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (33)$$

Let  $\alpha = U\delta$ . Then we obtain the equation

$$\alpha^T \Sigma \alpha = \delta^T U^T \Sigma U \delta = \delta^T D(\lambda) \delta = \sigma_Y^2 \quad (34)$$

In scalar form we have the familiar equation (i.e. for the linear operator in  $\mathbb{R}^2$ )

$$\lambda_1 \delta_1^2 + \lambda_2 \delta_2^2 = \sigma_Y^2 \quad (35)$$

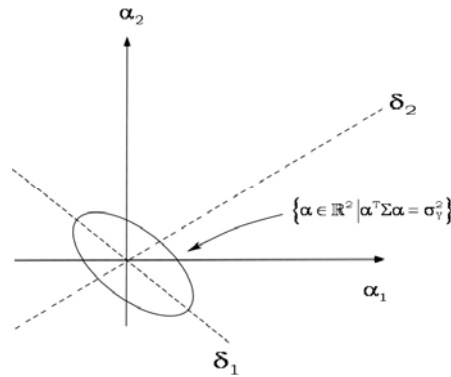
We interpret the transformation  $\alpha = U\delta$  as a transformation of the coordinate system so the graph of

$$\alpha^T \Sigma \alpha = \sigma_Y^2 \quad (36)$$

is not changed. However the alteration makes the equation so simple that the nature of its graph may be determined by inspection. If  $\lambda_1 \neq 0, \lambda_2 \neq 0$ , and  $\sigma_Y^2 \neq 0$  then

$$\frac{\delta_1^2}{\sigma_Y^2/\lambda_1} + \frac{\delta_2^2}{\sigma_Y^2/\lambda_2} = 1 \quad (37)$$

If  $\sigma_Y^2/\lambda_j > 0$  for  $j=1, 2$  then (38) is the equation of an ellipse. If  $\lambda_1 = \lambda_2$  then (39) is a circle. If  $\lambda_1$  and  $\lambda_2$  are opposite in sign then (40) is a hyperbola.



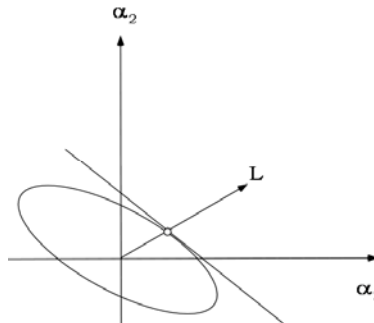
The characteristic roots  $\lambda \in \mathbb{R}^n$  of  $\Sigma$  are all positive if and only if the quadratic form  $\alpha^T \Sigma \alpha$  is positive definite and the quadratic form is positive definite if and only if the leading principal minors of the matrix of the form are all positive, i.e.

$$\sigma_{11} > 0, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} > 0, \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} > 0, \dots, \begin{pmatrix} \sigma_{11} & \dots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \dots & \sigma_{nn} \end{pmatrix} > 0$$

The dominant  $\alpha \in \mathbb{R}^n$  are those which minimize  $\sigma_Y^2$  subject to a fixed value of  $\mu_Y$ . Hence, we have the problem

$$\text{Minimize } \alpha^T \Sigma \alpha$$

$$\text{Subject to } w(1 + x_0) + \alpha(\mu + ex_0) = \mu_Y^0$$



Or the LaGrange function

$$L(\alpha, \delta) = \alpha^T \sum \alpha - \delta (w(1 + x_0) + \alpha(\mu - ex_0) - \mu_Y^0) \quad (41)$$

The first order conditions are

$$\begin{aligned} D_1 L &= 2\alpha_1 \sigma_{11} + 2\alpha_2 \sigma_{12} - \delta(\mu_1 - x_0) = 0 \\ D_2 L &= 2\alpha_1 \sigma_{12} + 2\alpha_2 \sigma_{22} - \delta(\mu_2 - x_0) = 0 \\ D_\delta L &= w(1 + x_0) + \alpha(\mu - ex_0) - \mu_Y^0 = 0 \end{aligned} \quad (42)$$

In matrix notation the first two conditions may be rewritten as

$$\sum \alpha = \frac{1}{2} \delta (\mu - ex_0) \quad (43)$$

These conditions may as be expressed as

$$-\frac{\alpha_1 \sigma_{11} + \alpha_2 \sigma_{12}}{\alpha_1 \sigma_{12} + \alpha_2 \sigma_{22}} = -\frac{\mu_1 - x_0}{\mu_2 - x_0} \quad (44)$$

which is the familiar tangency condition. The LHS is the slope of the ellipse and the RHS is the slope of the constant mean line. Since  $f(\alpha) = \alpha^T \sum \alpha$  is homogeneous of degree two the set of points  $\alpha \in \mathbb{R}^N$  which satisfy (45) or (46) is the linear function L. This result may also be noted by observing that

$$\frac{t\alpha_1 \sigma_{11} + t\alpha_2 \sigma_{12}}{t\alpha_1 \sigma_{12} + t\alpha_2 \sigma_{22}} = \frac{\alpha_1 \sigma_{11} + \alpha_2 \sigma_{12}}{\alpha_1 \sigma_{12} + \alpha_2 \sigma_{22}} \quad (47)$$

Hence the investor holds the risky assets in fixed proportions, i.e.  $\alpha_2 = k\alpha_1$ . Thus we may form a composite risky asset by letting

$$\sum_{j=1}^N \alpha_j x_j = \alpha_1 x_I \quad (48)$$

where

$$x_1 = \sum_{j=1}^N \frac{\alpha_j}{\alpha_1} x_j \quad (49)$$

and

$$\sum_{j=1}^N \frac{\alpha_j}{\alpha_1} = 1 \quad (50)$$

defines  $\alpha_1$ . Then since  $\alpha_2 = k\alpha_1$  we obtain

$$x_1 = \frac{1}{1+k} x_1 + \frac{k}{1+k} x_2 \quad (51)$$

Then

$$Y = w(1 + x_0) + \alpha_1(x_1 - x_0) \quad (52)$$

and it follows that

$$\mu_Y = w(1 + x_0) + \frac{\mu_1 - x_0}{\sigma_1} \sigma_Y \quad (53)$$

is the trading line.

## References

Tobin, J. (1958). "Liquidity Preference as Behavior Toward Risk." Review of Economic Studies **25**(2): 65-86.