Notes on Tobin's

Liquidity Preference as Behavior toward Risk

By

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Revised subsequently

Tobin (Tobin 1958) considers a portfolio model in which there is one safe and one risky asset. The net rate of return is X_0 on the safe asset and X_1 on the risky asset.

A portfolio consists of a dollar amount α_0 invested in the safe asset and an amount α_1 invested in the risky asset where $\alpha_0 + \alpha_1 = w$ and w is the investor's initial wealth. Let Y be a random variable denoting the return on the portfolio. Then

$$\begin{aligned} Y &= \alpha_0 (1 + x_0) + \alpha_1 (1 + x_0) \\ &= w(1 + x_0) + \alpha_1 (x_1 - x_0) \end{aligned} \tag{1}$$

and

$$\begin{aligned} EY &\equiv \mu_Y \\ &= w(1 + x_0) + \alpha_1 (\mu_1 - x_0) \\ \text{Var } Y &\equiv \sigma_Y^2 \\ &= \alpha_1^2 \text{Var } x_1 \\ \sigma_Y &= \alpha_1 \sigma_1 \end{aligned} \tag{2}$$

The amount α_1 determines both μ_Y and σ_Y. The terms on which an investor can obtain a greater expected return at the expense of more risk is

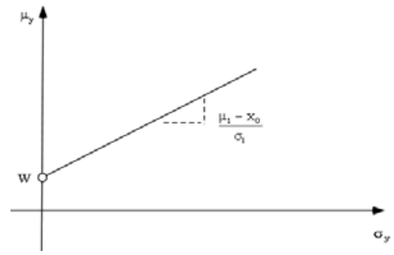
$$\mu_Y = w(1 + x_0) + \frac{\mu_1 - x_0}{\sigma_1} \sigma_Y \tag{3}$$

where $0 \le \sigma_Y \le w\sigma_1$. (3) is the investor's trading line. The investor holds the amount $\alpha_1 = \sigma_Y / \sigma_1$ of the risky asset.

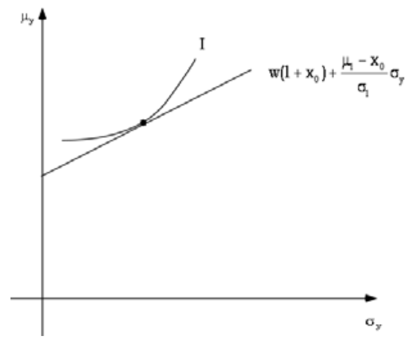
Let $\alpha_1 = \lambda_1 w$. Then

$$\begin{aligned} \mu_Y - w &= wx_0 + \frac{\mu_1 - x_0}{\sigma_1} \lambda_1 w \sigma_1 \\ \frac{\mu_Y - w}{w} &= x_0 + \frac{\mu_1 - x_0}{\sigma_1} \lambda_1 \sigma_1 \end{aligned} \tag{4}$$

This is called the capital market line where $\lambda_1 \sigma_1$ is the standard deviation per unit of wealth held in the risky asset and $\frac{\mu_1 - x_0}{\sigma_1}$ is sometimes called market price of risk.



The investor is assumed to have preferences which rank all (σ_Y, μ_Y) pairs and these preferences can be represented by indifference curves. Then the individual maximizes expected utility subject to selecting a pair (σ_Y, μ_Y) on the trading line.



There are two rationales for supposing that

$$E\mu(Y) = f(\mu_Y, \sigma_Y) \tag{5}$$

The first is that $Y \sim N(\mu_Y, \sigma_Y)$ Let $g(y; \mu_Y, \sigma_Y)$ be the density function of Y. Then

$$\begin{aligned} E\mu(Y) &= \int_{-\infty}^{\infty} \mu(y) g(y; \mu_Y, \sigma_Y) dy \\ &= f(\mu_Y, \sigma_Y) \end{aligned} \tag{6}$$

and the shape of the indifference curve can be inferred from the shape of μ. To demonstrate this we perform the following transformation.

Let

$$\begin{aligned} Y &= \Phi(z) \equiv \mu_Y + \sigma_Y z \\ z &= \phi(Y) \equiv \frac{Y - \mu_Y}{\sigma_Y} \end{aligned} \tag{7}$$

i.e. $\phi = \Phi^{-1}$. Then

$$
\begin{aligned}
E\mu(Y) &= E\mu(\Phi(z)) \\
&= \int_{-\infty}^{\infty} \mu(\Phi(z))h(z;0,1)dz \\
&= f(\mu_Y, \sigma_Y)
\end{aligned} \tag{8}
$$

where $z \sim N(0,1)$ and h is the standard normal density. Let $D_1 f$ and $D_2 f$ denote the partial derivatives of f with respect to μ_Y and σ_Y respectively. Then

$$
\begin{aligned}
D_1 f &= \int_{-\infty}^{\infty} \mu' h \\
D_2 f &= \int_{-\infty}^{\infty} \mu' zh
\end{aligned} \tag{9}
$$

Since f is constant on the indifference curve

$$
0 = df = D_1 f d\mu_Y + D_2 f d\sigma_Y \tag{10}
$$

or equivalently

$$
\begin{aligned}
\frac{d\mu_Y}{d\sigma_Y} &= -\frac{D_2 f}{D_1 f} \\
&= -\frac{\int_{-\infty}^{\infty} zu'(\Phi(z))h(z)}{\int_{-\infty}^{\infty} u'(\Phi(z))h(z)}
\end{aligned} \tag{11}
$$

Claim: if u' > 0 and u" < 0 then $d\mu_Y / d\sigma_Y > 0$.

Proof. Since the denominator is clearly positive, it suffices to show that u"<0 implies

$$
N \equiv \int_{-\infty}^{\infty} zu'(\Phi(z))h(z) < 0 \tag{12}
$$

Note that

$$
N = \lim_{a \to \infty} \left\{ \int_{-a}^{0} zu'h + \int_{0}^{a} zu'h \right\} \tag{13}
$$

where h is symmetric, i.e. h(t) = h(-t). Let z = -t. Then

$$
\begin{aligned}
&\int_{-a}^{0} zu'(\Phi(z))h(z)dz = \\
&\int_{a}^{0} -tu'(\Phi(-t))h(-t)(-dt)
\end{aligned} \tag{14}
$$

and

$$
\begin{aligned}
N &= \lim_{a \to \infty} \left\{ -\int_{0}^{a} tu'(\Phi(-t))h(-t)dt + \int_{0}^{a} tu'(\Phi(t))h(t)dt \right\} \\
&= \lim_{a \to \infty} \left\{ \int_{0}^{a} t\left[u'(\Phi(t)) - u'(\Phi(-t))\right]h(t) \right\} < 0
\end{aligned} \tag{15}
$$

Since $u'(\Phi(-t)) > u'(\Phi(t))$.

Comparative Statistics

Suppose that u is an increasing concave function and that $X_1 \sim N(\mu_1, \sigma_1^2)$. Now suppose μ_1 changes. Then we want to determine what effect this will have on the investor's optimal choice of α_1.

Recall that

$$\begin{aligned}\Phi(z) &= \mu_Y + \sigma_Y z \\ &= w(1+x_0) + \alpha_1(\mu_1 - x_0) + \alpha_1\sigma_1 z\end{aligned} \tag{16}$$

and the investor's problem is to select α_1 to maximize $\int_{\mathbb{R}} u(\Phi(z))h(z)$. The first order condition is

$$\int_{\mathbb{R}} u'(\Phi(z))(\mu_1 - x_0 + \sigma_1 z)h(z) = 0 \tag{17}$$

or equivalently

$$\frac{\mu_1 - x_0}{\sigma_1} = -\frac{\int_{\mathbb{R}} u'zh}{\int_{\mathbb{R}} u'h} \tag{18}$$

Now let the function $F : D \to \mathbb{R}, D \subset \mathbb{R}^2$ be defined by

$$F(\mu_1, \alpha_1) = \int_{\mathbb{R}} u'[\mu_1 - x_0 + \sigma_1 z]h \tag{19}$$

Since $D_2F < 0$, it follows by the Implicit Function Theorem that there exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $F(\mu_1, f(\mu_1)) = 0$ and $f' = -D_1F/D_2F$.

Note that $f' \gtrless 0$ as $D_1F \gtrless 0$.

$$\begin{aligned}D_1F &= \int_{\mathbb{R}} \{u' + u''\alpha_1[\mu_1 - x_0 + \sigma_1 z]\}h \\ &= \int_{\mathbb{R}} u'h + \alpha_1\int_{\mathbb{R}} u''[\mu_1 - x_0 + \sigma_1 z]h > 0\end{aligned} \tag{20}$$

if the second integral on the RHS is non-negative. To determine the sign let $a \equiv -u''/u'$ denote the measure of absolute risk aversion. Assume that $a' \le 0$ and let z_0 be implicitly defined by $\mu_1 - x_0 + \sigma_1 z_0 = 0$ (i.e. $z_0 = -\frac{\mu_1 - x_0}{\sigma_1}$).

Then

$$-\frac{u''(\Phi(z_0))}{u'(\Phi(z_0))} \le a(\Phi(z_0)), z \ge z_0 \tag{21}$$

or

$$u''(\Phi(z_0)) \geq -a(\Phi(z_0))u'(\Phi(z_0)) \tag{22}$$

which in turn yields

$$u''(\Phi(z_0))(\mu_1 - x_0 + \sigma_1 z) \geq -a(\Phi(z_0))u'(\Phi(z))(\mu_1 - x_0 + \sigma_1 z) \tag{23}$$

for $z \geq z_0$. A similar argument shows that (23) holds for $z < z_0$. Hence, given $a' \leq 0$,

$$\int_{\mathbb{R}} u''(\Phi(z))(\mu_1 - x_0 + \sigma_1 z)h \geq -a(\Phi(z_0))\int_{\mathbb{R}} u'(\Phi(z))(\mu_1 - x_0 + \sigma_1 z)h \tag{24}$$

but the RHS is zero at the optimal α_1 . Hence, $D_1F \geq 0$ or equivalently $f'(\mu_1) > 0$. Note that in Tobin's model (i.e. where $x_0 \equiv 0$ and so the safe asset is money) $f'(\mu_1) > 0$ also implies that the demand for money is a decreasing function of the expected return on the risky asset, which he calls the rate of interest.

Next consider a change in the riskness of asset one. Let $G : D \to \mathbb{R}, D \subset \mathbb{R}^2$, be defined as

$$G(\sigma_1, \alpha_1) = \int_{\mathbb{R}} u'(\Phi(z))(\mu_1 - x_0 + \sigma_1 z)h \tag{25}$$

Then as above there exist a function g such that $G(\sigma_1, g(\sigma_1)) = 0$ and $g' = -\dfrac{D_1G}{D_2G} \gtrless 0$ as $D_1G \gtrless 0$.

$$\begin{aligned} D_1G &= \int_{\mathbb{R}} \left[u'z + \alpha_1 u''z(\mu_1 - x_0 + \sigma_1 z)\right]h \\ &= \int_{\mathbb{R}} u'zh + \alpha_1 \int_{\mathbb{R}} u''z(\mu_1 - x_0 + \sigma_1 z)h \end{aligned} \tag{26}$$

Recall that the first integral on the RHS is negative and so $D_1G < 0$ if the second integral on the RHS is non-positive. Since

$$\begin{aligned} &\int_{\mathbb{R}} u''z(\mu_1 - x_0 + \sigma_1 z)h = \frac{1}{\sigma_1}\int_{\mathbb{R}} u''\sigma_1 z(\mu_1 - x_0 + \sigma_1 z)h = \\ &\frac{1}{\sigma_1}\int_{\mathbb{R}} u''\left[(\mu_1 - x_0) + \sigma_1 z - (\mu_1 - x_0)\right](\mu_1 - x_0 + \sigma_1 z)h = \\ &\frac{1}{\sigma_1}\left\{\int_{\mathbb{R}} u''(\mu_1 - x_0 + \sigma_1 z)^2 h - (\mu_1 - x_0)\int_{\mathbb{R}} u''(\mu_1 - x_0 + \sigma_1 z)h\right\} \end{aligned} \tag{27}$$

It follows that $a' \leq 0$ and $\mu_1 - x_0 \geq 0$ suffice to show $D_1G < 0$.

Model with Many Risky Assets

Suppose there are N risky assets. Let $\alpha = (\alpha_1, \ldots., \alpha_N) \in \mathbb{R}^N$ where $\alpha_j, j = 1, \ldots, N$ is the dollar amount held of risky asset j_1 and α_0 is defined as before. Let x_j denote the random rate of return on asset j. Then

$$\begin{aligned} Y &= \alpha_0 (1 + x_0) + \sum_{j=1}^{N} \alpha_j (1 + x_j) \\ &= w(1 + x_0) + \sum_{j=a}^{N} \alpha_j (x_j - x_0) \end{aligned} \tag{28}$$

since

$$\alpha_0 = w - \sum_{j=1}^{N} \alpha_j$$

Let $$\sigma_{ij} = E(x_i x_j) - Ex_i Ex_j$$

Then

$$\mu_Y = w(1 + x_0) + \sum_{j=1}^{N} \alpha_j (\mu_j - x_0) \tag{29}$$

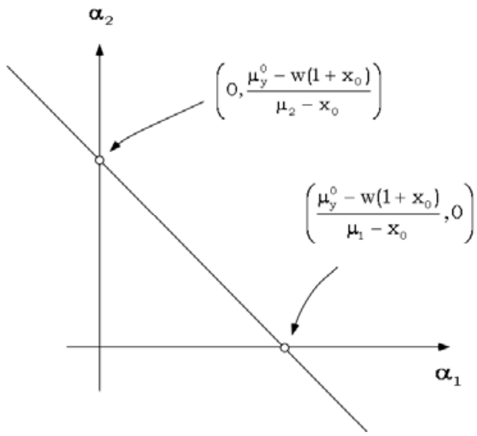
and

$$\begin{aligned} \sigma_Y^2 &= \mathrm{Var}\left(\sum_{j=1}^{N} \alpha_j x_j\right) \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \sigma_{ij} \end{aligned} \tag{30}$$

In the special case N=2

$$\begin{aligned} &\mathrm{Var}(\alpha_1 x_1 + \alpha_2 x_2) = \\ &E(\alpha_1 x_1 + \alpha_2 x_2 - (\alpha_1 \mu_1 + \alpha_2 \mu_2))^2 = \\ &E(\alpha_1 (x_1 - \mu_1) + \alpha_2 (x_2 - \mu_2))^2 = \\ &\alpha_1^2 E(x_1 - \mu_1)^2 + 2\alpha_1 \alpha_2 E[(x_1 - \mu_1)(x_2 - \mu_2)] + \alpha_2^2 E(x_2 - \mu_2)^2 = \alpha_1^2 \sigma_{11} + 2\alpha_1 \alpha_2 \sigma_{12} + \alpha_2^2 \sigma_{22} \end{aligned} \tag{31}$$

The set of points $\alpha \in \mathbb{R}^N$ for which μ_Y is constant is a hyperplane. In the case N=2 the $\{\alpha \in \mathbb{R}^2 | w(1 + x_0) + \alpha(\mu - ex_0) = \mu_Y^0\}$ where $e = (1, \ldots, 1) \in \mathbb{R}^N$ and $\mu = (\mu_1, \ldots, \mu_N)$ is shown



Let $\sum$ denote the variance-covariance matrix. Then $\sigma_Y^2 = \alpha^T \sum \alpha$. For N=2 the $\left\{\alpha \in \mathbb{R}^2 \middle| w(1+x_0) + \alpha(\mu - ex_0) = \mu_Y^0\right\}$ where $e = (1,\ldots,1) \in \mathbb{R}^N$ and $\mu = (\mu_1,\ldots,\mu_N)$ is shown

Let $\sum$ denote the variance-covariance matrix. Then $\sigma_Y^2 = \alpha^T \sum \alpha$. For N=2

$$\sum = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \tag{32}$$

Note $\sum$ is a real symmetric matrix and so there exists an orthogonal matrix U such that $U^T \sum U$ is a diagonal matrix whose diagonal elements are the characteristic roots of $\sum$. Since $\sum$ is a real symmetric matrix, all its characteristic roots are real. Let $U^T \sum U = D(\lambda)$ where $\lambda \in \mathbb{R}^n$ is the vector of characteristic roots, e. g.

$$D(\lambda) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{33}$$

Let $\alpha = U\delta$. Then we obtain the equation

$$\alpha^T \sum \alpha = \delta^T U^T \sum U\delta = \delta^T D(\lambda)\delta = \sigma_Y^2 \tag{34}$$

In scalar form we have the familiar equation (i.e. for the linear operator in $\mathbb{R}^2$)

$$\lambda_1\delta_1^2 + \lambda_2\delta_2^2 = \sigma_Y^2 \tag{35}$$

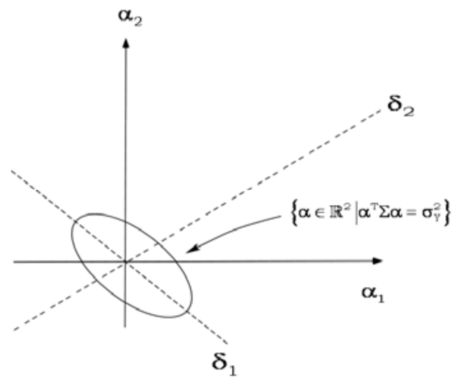
We interpret the transformation $\alpha = U\delta$ as a transformation of the coordinate system so the graph of

$$\alpha^T \sum \alpha = \sigma_Y^2 \tag{36}$$

is not changed. However the alteration makes the equation so simple that the nature of its graph may be determined by inspection. If $\lambda_1 \neq 0, \lambda_2 \neq 0$, and $\sigma_Y^2 \neq 0$ then

$$\frac{\delta_1^2}{\sigma_Y^2 / \lambda_1} + \frac{\delta_2^2}{\sigma_Y^2 / \lambda_2} = 1 \tag{37}$$

If $\sigma_Y^2 / \lambda_j > 0$ for j=1, 2 then (38) is the equation of an ellipse. If $\lambda_1 = \lambda_2$ then (39) is a circle. If λ_1 and λ_2 are opposite in sign then (40) is a hyperbola.



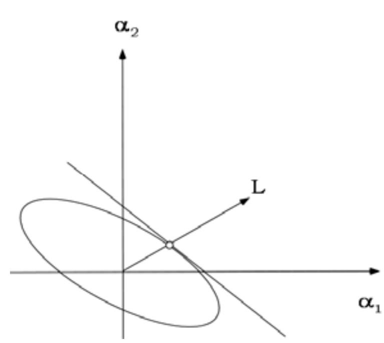
The characteristic roots $\lambda \in \mathbb{R}^n$ of $\sum$ are all positive if and only if the quadratic form $\alpha^T \sum \alpha$ is positive definite and the quadratic form is positive definite if and only if the leading principal minors of the matrix of the form are all positive, i.e.

$$\sigma_{11} > 0, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} > 0, \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} > 0, \ldots, \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{pmatrix} > 0$$

The dominant $\alpha \in \mathbb{R}^N$ are those which minimize σ_Y^2 subject to a fixed value of μ_Y. Hence, we have the problem

Minimize $\alpha^T \sum \alpha$

Subject to $w(1 + x_0) + \alpha(\mu + e x_0) = \mu_Y^0$



Or the LaGrange function

$$L(\alpha,\delta)=\alpha^{T}\sum\alpha-\delta\left(w(1+x_0)+\alpha(\mu-ex_0)-\mu_Y^0\right) \tag{41}$$

The first order conditions are

$$\begin{aligned} D_1L &= 2\alpha_1\sigma_{11}+2\alpha_2\sigma_{12}-\delta(\mu_1-x_0)=0 \\ D_2L &= 2\alpha_1\sigma_{12}+2\alpha_2\sigma_{22}-\delta(\mu_2-x_0)=0 \\ D_\delta L &= w(1+x_0)+\alpha(\mu-ex_0)-\mu_Y^0=0 \end{aligned} \tag{42}$$

In matrix notation the first two conditions may be rewritten as

$$\sum\alpha=\frac{1}{2}\delta(\mu-ex_0) \tag{43}$$

These conditions may as be expressed as

$$-\frac{\alpha_1\sigma_{11}+\alpha_2\sigma_{12}}{\alpha_1\sigma_{12}+\alpha_2\sigma_{22}}=-\frac{\mu_1-x_0}{\mu_2-x_0} \tag{44}$$

which is the familiar tangency condition. The LHS is the slope of the ellipse and the RHS is the slope of the constant mean line. Since $f(\alpha)=\alpha^{T}\sum\alpha$ is homogeneous of degree two the set of points $\alpha\in\mathbb{R}^N$ which satisfy (45) or (46) is the linear function L. This result may also be noted by observing that

$$\frac{t\alpha_1\sigma_{11}+t\alpha_2\sigma_{12}}{t\alpha_1\sigma_{12}+t\alpha_2\sigma_{22}}=\frac{\alpha_1\sigma_{11}+\alpha_2\sigma_{12}}{\alpha_1\sigma_{12}+\alpha_2\sigma_{22}} \tag{47}$$

Hence the investor holds the risky assets in fixed proportions, i.e. $\alpha_2=k\alpha_1$. Thus we may form a composite risky asset by letting

$$\sum_{j=1}^{N}\alpha_j x_j=\alpha_I x_I \tag{48}$$

where

$$x_I = \sum_{j=1}^{N} \frac{\alpha_j}{\alpha_I} x_j \qquad (49)$$

and

$$\sum_{j=1}^{N} \frac{\alpha_j}{\alpha_I} = 1 \qquad (50)$$

defines α_I. Then since $\alpha_2 = k\alpha_1$ we obtain

$$x_I = \frac{1}{1+k} x_1 + \frac{k}{1+k} x_2 \qquad (51)$$

Then

$$Y = w\left(1 + x_0\right) + \alpha_I \left(x_I - x_0\right) \qquad (52)$$

and it follows that

$$\mu_Y = w\left(1 + x_0\right) + \frac{\mu_I - x_0}{\sigma_I} \sigma_Y \qquad (53)$$

is the trading line.

References

Tobin, J. (1958). "Liquidity Preference as Behavior Toward Risk." Review of Economic Studies **25**(2): 65-86.