The Annuity Puzzle

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1. The Puzzle

Consumption-saving behavior has been studied and discussed by economists including Marshall and Fisher (Marshall 1920; Fisher 1930), but until Yaari (Yaari 1965) the question of how a consumer should optimally allocate her limited resources over an uncertain lifetime had not been carefully addressed. In his seminal piece, Yaari extends the analysis of optimal consumption plans by maximizing an investor's expected utility over a random time horizon. In particular, Yaari shows that investors without bequest motives will find it optimal to completely annuitize their savings. In view of this result, Friedman and Warshawsky note:

> It is startling, at least for economists who view consumption-saving behavior within the framework of the familiar life-cycle model, to confront the fact that in the United States few individuals purchase life annuities. According to the life-cycle model, the chief principle governing individual saving behavior is the desire to smooth consumption patterns over one's lifetime, within the constraints imposed by limited lifetime resources.[1]

Indeed the full annuitization results or predictions have become known as the annuity puzzle because the life annuity markets in the United States and elsewhere are so thin.[2] The objective of this analysis is not to review[3] the many the attempts to solve the annuity puzzle but is to extend the expected utility paradigm to provide the economic foundations for the investigation of life insurance and life annuities. The working hypothesis here is that the foundations of life insurance and life annuities have not been adequately developed and that an altered paradigm that explains the demand for life insurance is also essential to understanding the anemic annuity market.

The annuity puzzle is demonstrated in section two in the context of the Fisher model. The individual may transfer money between dates using annuities, bonds and life insurance. As in the standard economic paradigm, the individual decision maker acts entirely in the pursuit of self interest. In this context the analysis reveals not only an annuity puzzle but also a life puzzle. While the first is a puzzle because the model predicts a robust annuity market in contradiction to observation, the second is a puzzle because the model predicts no life market also in

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[1] See p. 135 in Friedman, B. M. and M. J. Warshawsky (1990). "The cost of annuities: implications for saving behavior and bequests." Quarterly Journal of Economics **105**(1): 135-154.

[2] See Johnson, R. W., L. E. Burman, et al. (2004). Annuitized wealth at older ages: Evidence from the health and retirement study. Final Report to the Employee Benefits Security Administration. Washington, DC, US Department of Labor. and James, E. and X. Song (2001). Annuity markets around the world: money's worth and risk intermediation, CeRP Working Paper 16/0.

[3] The literature on the annuity puzzle is large and growing. See Brown, J. R., O. S. Mitchell, et al. (2001). The Role of Annuity Markets in Financing Retirement, MIT. and two recent books that provide commentary on the annuity markets and puzzle, i.e., Cannon, E. and I. Tonks (2008). Annuity Markets. Oxford, Oxford University Press, Sheshinski, E. (2008). The Economic Theory of Annuities. Princeton, Princeton University Press. Also see Brown, J. R. (2004). "Life Annities and Uncertain Lifetimes." NBER Reporter: Research Summary Spring 2004, from http://www.nber.org/reporter/spring04/brown.html#N_1_. for a summary.

contradiction to observation. Indeed the individual who acts entirely in the pursuit of self interest would never purchase life insurance. The dual puzzles are either puzzles or suggest the necessity of a paradigm change. The paradigm change is addressed in section three. The change is similar to the bequest motive that has received so much attention in the literature. Simulation results in the literature, however, provide contradictory evidence of the importance of the bequest motive, e.g. see (Davidoff, Brown et al. 2005; Inkmann, Lopes et al. 2007; Purcal and Piggott 2008; Lockwood 2009). Lockwood has recently argued that the marginal value of bequests may be sufficient to eliminate the demand for annuities and demonstrated with simulations. The paradigm in section three is used to develop a model in which the welfare of a significant other affects all the choices made by the decision maker and the model demonstrates that any load on the annuity will eliminate long positions in the instrument. In section four, annuity provider insolvency risk is introduced; the analysis shows that the insolvency risk even with fairly priced annuities will end the dominance of annuities. In section five, a health risk is introduced and the analysis suggests that an illiquid annuity will not be included in the individual's portfolio. Section six concludes.

2. The Classic Economic Paradigm

The annuity puzzle is demonstrated in Davidoff et. al. (Davidoff, Brown et al. 2005) but not resolved. The special assumptions that are necessary in Yaari (Yaari 1965), however, are relaxed by Davidoff *et. al.* and the authors show that the life annuity dominates the bond contract. We provide an intuitive sketch of that result here in an expected utility framework.

Consider an individual consumer/investor. The individual makes decisions at t = 0 and the receives the payoffs from those decisions at t = 1. For simplicity the dates t = 0 and 1 are referred to as *now* and *then*, respectively. The decisions concern a portfolio of assets that are consumed *then* if the individual survives. The individual survives the period with probability q. Let the portfolio include life annuities, bonds and life insurance. Suppose λ_a and λ_b represent the number of life annuities and bonds, respectively, held by the individual. Suppose the annuity provides a one dollar return *then* if the individual survives and zero otherwise for each annuity purchased *now*; let $p_a = (1-q)/(1+r)$ denote the annuity price *now* where $(1-q)$ is the probability of survival and where the equality holds if annuities are price fairly. Similarly, suppose the bond provides one dollar return *then* for each bond purchased *now* and let $p_b = 1/(1+r)$ denote the bond price *now* where the equality holds if the bond market is competitive. The life annuity differs from the bond instrument because it has a survival trigger. Also consider a life insurance contract. Suppose λ_l represents the number of life contracts. The life insurance provides one dollar then for each life contract purchased if the insured dies and zero otherwise; let $p_l = q/(1+r)$ denote the price of the life insurance contract *now* where the equality holds if the life insurance is price fairly. The individual investor selects a life insurance contract and a portfolio of life annuities and bonds that determine consumption *now* and *then*, i.e., (c_0, c_1), as follows:

$$c_0 = w - p_a\lambda_a - p_b\lambda_b - p_l\lambda_l$$

where *w* represents the wealth *now* and

$$c_1 = \begin{cases} 0 & q \\ \lambda_a + \lambda_b & 1-q \end{cases}$$

where consumption *then* depends upon the realized state; the states are death with probability q and survival with probability $(1-q)$. The life insurance contract yields λ_l dollars in the event of death and zero otherwise. The individual, however, cannot consume the λ_l dollars; it is a payment that goes to the beneficiary. Hence, consumption *then* in the death event is zero. The life annuity yields λ_a dollars if the individual survives and zero otherwise. The bond yields the λ_b dollars whether the individual survives or not but death precludes the consumption of those dollars. Let u denote the utility of consumption *now* and *then* so that the expected utility is

$$\begin{aligned} F(\lambda) = {} & u\left(w - p_a\lambda_a - p_b\lambda_b - p_l\lambda_l, 0\right)q \\ & + u\left(w - p_a\lambda_a - p_b\lambda_b - p_l\lambda_l, \lambda_a + \lambda_b\right)(1-q) \end{aligned} \tag{1}$$

The derivatives with respect to the life annuity, bond and life insurance contracts are the following:

$$D_1F \equiv \frac{\partial F}{\partial \lambda_a} = -D_1u\ p_a\ q - D_1u\ p_a\,(1-q) + D_2u\,(1-q) \tag{2}$$

$$D_2F \equiv \frac{\partial F}{\partial \lambda_b} = -D_1u\ p_b\ q - D_1u\ p_b\,(1-q) + D_2u\,(1-q) \tag{3}$$

and

$$D_3F \equiv \frac{\partial F}{\partial \lambda_l} = -D_1u\ p_l\ q - D_1u\ p_l\,(1-q) < 0 \tag{4}$$

Observe that the inequality in (4) shows quite clearly that the optimal life insurance benefit is zero, or equivalently, no life insurance is demanded. Although seldom noted in this annuity literature, this should come as no surprise since the classic economic agent prefers more to less. Hence, the classic economic paradigm has no explanation for life insurance.

Next suppose that λ_a and λ_b are both positive and consider the derivative of the expected utility F in the direction $v = (1, -1, 0)$; let $D_v F$ denote the derivative. Moving in this direction amounts to annuitizing savings

$$\begin{aligned} D_v F &= D_1 F - D_2 F \\ &= -D_1 u\; p_a\, q - D_1 u\; p_a\,(1-q) + D_2 u\,(1-q) \\ &\quad + D_1 u\; p_b\, q + D_1 u\; p_b\,(1-q) - D_2 u\,(1-q) \qquad (5) \\ &= \left(p_b - p_a\right)\left\{D_1 u\, q + D_1 u\,(1-q)\right\} \\ &> 0 \end{aligned}$$

The last term in brackets on the right hand side is the expected marginal utility of consumption *now* and is positive since more is preferred to less. The inequality follows if and only if the bond price exceeds the annuity price. Hence annuitizing savings or equivalently decreasing bond holdings and increasing life annuity holdings increases expected utility. Equivalently, the investors select a portfolio with only life annuity contracts. This inequality forms the basis for the annuity puzzle because it shows in a very simple setting that the annuity dominates the bond.

Indeed, the classic paradigm yields not only an annuity puzzle but also a life puzzle. The unambiguous sign of the derivative in (4) demonstrates the basis for the life puzzle while the unambiguous sign of the derivative in (5) demonstrates the basis for the annuity puzzle. In the annuity puzzle, the theory predicts a strong annuity market and the puzzle is why we do not see strong annuity markets anywhere in the world. In the life puzzle, the theory predicts no or at least a weak life markets and the puzzle is why we see strong life markets.

The inequality in (5) continues to hold even in the presence of premium loading on the annuities. Let δ denote the loading factor so that the return on the annuity is $R_a = \left(1-\delta\right)\left(1+r\right) / \left(1-q\right)$ and its price is

$$p_a = \frac{\left(1-q\right)}{\left(\left(1-\delta\right)\left(1+r\right)\right)} \qquad (6)$$

With this loading (5) continues to hold if and only if the loading factor is no greater than the probability of death, i.e.,

$$\begin{aligned} p_b - p_a &= \frac{1}{1+r} - \frac{1-q}{(1-\delta)(1+r)} \\ &= p_b\left(1 - \frac{1-q}{1-\delta}\right) \\ &= p_b \frac{q-\delta}{1-\delta} \\ &\geq 0 \end{aligned} \tag{7}$$

for $\delta \leq q$. Of course, even with the standard paradigm the annuity demand disappears if the load exceeds the probability of death.

3. An Altered Paradigm

Consider a different approach to explaining behavior. Suppose the individual can be described as an altruist.[4] An economy may be characterized not only as a set of contracts and a set of actions but also as a set of individuals. It is this last set that is so often an after-thought in theory. In the classic paradigm, the individual prefers more to less but that preference only includes goods and services. Here the more is preferred to less is also employed but extends to the well being of others as well as goods and services.

The approach adopted here is like but not equivalent to what Yaari called the bequest motive; in the bequest model the individual gains utility from dollars bequeathed in the death event; the individual in the bequest model is otherwise completely self centered as in the classic economic paradigm. In the model developed here the individual gains satisfaction or utility from the increased utility of a significant other whether that consumption occurs *now* or *then* or in the survival or death events. The individual is also sensitive to the risk aversion of the significant other. The individual decision maker here balances lost utility *now* versus gained utility *then* for self and significant other. In these respects it differs from the bequest model.

Most of the remaining analysis will use the notion of altruism to describe behavior. It is one seemingly obvious but neglected motivation for life insurance and has implications for life annuities. Also, to simplify, suppose the individual has one significant other and let that significance mean that the individual in question has a utility that increases with that of at least one the significant other. For simplicity we will suppose there is only one significant other with utility v. It follows that our individual has preferences defined on (c, v) where v is the significant other's utility. We could muddy the waters by supposing that the utility of the significant other

[4] The individual in the last section was egocentric. One could also, for some purposes, describe an individual as a sadistic if that individual's utility increased as another's decreased.

has a utility that also depends on u; we will not. Rather we will suppose that the significant other is entirely selfish just as all agents in the standard economic paradigm.

The addition of altruism does add the obvious element and motivation for life insurance. Given the existence of a single life annuity as well as other instruments such as bonds and stocks, the question becomes one of whether the individual has an incentive to include the life annuity, bonds and life insurance in a portfolio.

The altruistic individual will be shown to have a demand for life insurance. The individual choice in the last section is the standard paradigm or equivalently the egocentric case. The egocentric individual demands no life insurance but does demand life annuities and those life annuities dominate bonds, i.e., see (4) and (5), respectively. In the altruistic case the consumption becomes somewhat more complex. The investor consumes

$$c_{0i} = (1-\alpha)\left(w - p_a\lambda_a - p_b\lambda_b - p_l\lambda_l\right) \tag{8}$$

now, where $(1-\alpha)$ is the portion of initial wealth retained and α is the portion given to the significant other. The significant others consumption *now* is

$$c_{0s} = \alpha\left(w - p_a\lambda_a - p_b\lambda_b - p_l\lambda_l\right) \tag{9}$$

In this framework the significant other does not make an independent consumption decision but this is just a simplification.

If the investor divides w and bears the investment costs then those decisions would not affect the consumption *now* of the significant other.[5] This would, *ceteris paribus*, make the investments more attractive to the significant other. On the other hand if the investor splits the net then each decision affects the significant other's consumption *now* and so balances the benefits of the investment against the costs. As a first pass it seems appropriate to have the investor split the net amount so that consumption *now* for the individual and significant other is as shown in (8) and (9).

Next, consider the consumption *then* for the pair. Recall that for simplicity the significant other is assumed to not die while the individual dies with probability q and selects a death benefit λ_l where $0 \le \lambda_l \le w$. The second inequality is just for the convenience of calling $\lambda_l = w$ full insurance. Recall also that the income endowment assumed here is (w, 0). There are several reasons for this form. One is to emphasize that the second period in the model is retirement and to generate income in retirement it must be moved from *now* to *then* with a financial instrument.

[5]It is also possible to assume that the individual bears all the costs of building the portfolio and splits the net wealth *now* between himself and the significant other. This is pursued and will be noted when there are important differences in results.

All three financial instruments considered in this initial version of the model will move money forward to one or more members of interest. Another reason for the income endowment is to simplify the financial decisions to a choice of three variables. The life insurance moves dollars to the beneficiary or equivalently significant other in the next period if the individual dies. It does this at a cost *now* of $p_l = q/(1+r)$ per dollar in benefits; the equality holds if the insurance is price fairly. The life annuity moves dollars forward also but in this case only if the individual survives; the cost *now* of each dollar moved forward with the life annuity is $p_a = (1-q)/(1+r)$; again, the equality only holds if the annuity is priced fairly. The bond moves dollars forward without contingency in this initial version of the model; the cost of moving dollars forward with the bond instrument is $p_b = 1/(1+r)$ per dollar. It follows that consumption *then* for the individual is

$$c_{1i} = \begin{cases} c_{1id} & q \\ c_{1il} & 1-q \end{cases} = \begin{cases} 0 & q \\ (1-\alpha)\left(\lambda_a + \lambda_b\right) & 1-q \end{cases} \tag{10}$$

and for the significant other is

$$c_{1s} = \begin{cases} c_{1sd} & q \\ c_{1sl} & 1-q \end{cases} = \begin{cases} \lambda_b + \lambda_l & q \\ \alpha\left(\lambda_a + \lambda_b\right) & 1-q \end{cases} \tag{11}$$

Let $\lambda = \left(\lambda_a, \lambda_b, \lambda_l\right)$ and $p = \left(p_a, p_b, p_l\right)$. It follows that the individual's expected utility is $H(\lambda; p, q)$ where

$$H(\lambda) = u\left(c_{0i}, c_{1id}, v(c_{0s}, c_{1sd})\right)q + u\left(c_{0i}, c_{1il}, v(c_{0s}, c_{1sl})\right)(1-q) \tag{12}$$

Next consider how the optimum conditions here compare with that of the classic paradigm, i.e., the selfish or egocentric behavior in the last section. The first order condition with respect to life insurance is

$$D_3H \equiv \frac{\partial H}{\partial \lambda_l}$$

$$= -p_l \left\{ \left\{ D_1u\left(1-\alpha\right) + D_3u\, D_1v\,\alpha \right\} q + \left\{ D_1u\left(1-\alpha\right) + D_3u\, D_1v\,\alpha \right\} \left(1-q\right) \right\}$$

$$+ D_3u\, D_2v\, q \tag{13}$$

$$= -p_l\, M + D_3u\, D_2v\, q$$

$$= 0$$

where $M \equiv \left\{ \left\{ D_1u\left(1-\alpha\right) + D_3u\, D_1v\,\alpha \right\} q + \left\{ D_1u\left(1-\alpha\right) + D_3u\, D_1v\,\alpha \right\} \left(1-q\right) \right\}$ is the expected marginal utility of consumption now. Note that the first term in (13) mimics the marginal expected utility *now* in the previous section and says that the expected marginal utility of consumption *now* is negative since insuring reduces dollars *now* for the individual and significant other. The significant other, as beneficiary, receives dollars *then* in the death event and so the second term in (13) is positive. Also, note that if $D_3u > 0$ and $D_2v(c_{0s}, 0) = \infty$ then some life insurance is demanded. It is, of course, the second term that is missing in the classic paradigm or equivalently the purely selfish version of the model.

Considering the boundary condition for life insurance is interesting but more is necessary. Is the demand downward sloping? How does the demand change with an increase in mortality? Hence, consider the first order conditions for the life annuity and bond:

$$D_1 H \equiv \frac{\partial H}{\partial \lambda_a}$$

$$= \left\{ D_1 u \left[(1-\alpha)(-p_a) \right] + D_3 u\, D_1 v \left[\alpha(-p_a) \right] \right\} q$$

$$+ \left\{ D_1 u \left[(1-\alpha)(-p_a) \right] + D_2 u (1-\alpha) + D_3 u\, D_1 v \left[\alpha(-p_a) \right] + D_3 u\, D_2 v\, \alpha \right\} (1-q)$$

$$= \left\{ D_1 u \left[(1-\alpha)(-p_a) \right] + D_3 u\, D_1 v \left[\alpha(-p_a) \right] \right\} q$$

$$+ \left\{ D_1 u \left[(1-\alpha)(-p_a) \right] + D_3 u\, D_1 v \left[\alpha(-p_a) \right] \right\} (1-q)$$

$$+ \left\{ D_2 u (1-\alpha) + D_3 u\, D_2 v\, \alpha \right\} (1-q)$$

$$= -p_a M + \left\{ D_2 u (1-\alpha) + D_3 u\, D_2 v\, \alpha \right\} (1-q) \tag{14}$$

$$D_2 H \equiv \frac{\partial H}{\partial \lambda_b}$$

$$= \left\{ D_1 u \left[(1-\alpha)(-p_b) \right] + D_3 u\, D_1 v \left[\alpha(-p_b) \right] \right\} q$$

$$+ \left\{ D_1 u \left[(1-\alpha)(-p_b) \right] + D_2 u (1-\alpha) + D_3 u\, D_1 v \left[\alpha(-p_b) \right] + D_3 u\, D_2 v\, \alpha \right\} (1-q)$$

$$= \left\{ D_1 u \left[(1-\alpha)(-p_b) \right] + D_3 u\, D_1 v \left[\alpha(-p_b) \right] \right\} q$$

$$+ \left\{ D_1 u \left[(1-\alpha)(-p_b) \right] + D_3 u\, D_1 v \left[\alpha(-p_b) \right] \right\} (1-q)$$

$$+ D_3 u\, D_2 v\, q + \left\{ D_2 u (1-\alpha) + D_3 u\, D_2 v\, \alpha \right\} (1-q)$$

$$= -p_b M + D_3 u\, D_2 v\, q + \left\{ D_2 u (1-\alpha) + D_3 u\, D_2 v\, \alpha \right\} (1-q) \tag{15}$$

Now as in the previous section suppose the individual considers annuitizing savings; that is equivalent to moving one dollar from the bonds to the life annuity and then continuing to move in that direction v = (1, - 1, 0). The expected marginal utility in the direction v is

$$\begin{aligned} D_v H &= D_1 H - D_2 H \\ &= \left(p_b - p_a \right) M - D_3 u \, D_2 v \, q \end{aligned} \tag{16}$$

The first term on the RHS of (16) represent the expected marginal utility *now* of the move in the direction $v = (1, -1, 0)$ and the term is positive if it costs less to buy one dollar of income *then* with a life annuity than it does with a bond; if so then there is an increase in expected marginal utility *now* for the individual and significant other. The first term is similar to the derivative in the direction $v = (1, -1, 0)$ in the classic economic paradigm. Now, however, the expected marginal utility is also positive because the individual's utility increases with that of the significant other. The second term on the RHS of (16) is new and is due to altruism. Rather than having an unambiguously positive expected marginal utility in the direction *v* as we did in the classic economic paradigm, now we have a negative component since dollars invested in life annuities do not provide the significant other with a payoff in the event of the individual's death. Indeed, if the price of the life annuity is close to the bond price due to loading charges and a risk premium then the second term may dominate and, in the limit, yield no investment in the life annuity; Lockwood notes

> For people without bequest motives, increasing consumption at the expense of bequests is a free lunch that has huge welfare benefits in calibrated models. The benefits are so large because, without annuities, people who wish to smooth their consumption over time leave large bequests whether they value them or not. Kotlikoff and Spivak (1981) estimate that a 55-year-old man without a bequest motive consumes only about three-fourths of his wealth on average out of a desire to smooth consumption over time. Fully annuitizing his wealth using an annuity with a ten percent load would allow this man to consume 90 percent of his wealth on average, 15 percent more than he consumes without annuities. Of course, this increase in consumption comes at the expense of leaving smaller bequests |by fully annuitizing the individual leaves no bequest instead of leaving bequests worth one-fourth of his wealth on average| so people with bequest motives clearly gain less from this trade. Whereas someone without bequest motives would be willing to pay roughly 15 percent of his wealth for the opportunity to make this trade, someone who valued bequests at 50 cents on the dollar would be willing to pay roughly 2.5 percent of his wealth (15 - 0:50 - 25), just one-sixth as much.(Lockwood 2009)

We can inspect a boundary condition here. Suppose the individual has no bonds or life insurance, i.e., $(\lambda_b, \lambda_l) = (0, 0)$. Then the marginal utility $D_2 v$ in (16) is unbounded and so $D_v H < 0$; this says that the individual will find adding bonds rather than life annuities beneficial. Hence even with a smaller price for the life annuity than the bond, the analysis shows that the annuity does not dominate! Note that it is the altruistic element in the new paradigm that shows that annuities do not dominate debt even if life insurance is not included in the model.

The presence of life insurance in the model does simplify the analysis. Suppose that some life insurance is demanded so that the FOC in (13) holds with equality. Also consider an annuity with loading while maintaining an assumed no loading on other instruments. Then the annuity price may be expressed as it was in (6) and the derivative in the annuitizing direction becomes

$$D_v H = \left(p_b - p_a - p_l \right) M \tag{17}$$

since (13) yields $p_l = D_3 u \, D_2 v \, q / M$. Now observe that

$$\begin{aligned} p_b - p_a - p_l &= \left(\frac{1}{1+r} - \frac{1-q}{(1-\delta)(1+r)} \right) - \frac{q}{1+r} \\ &= p_b \left((1-q) - \frac{1-q}{(1-\delta)} \right) \\ &\le 0 \end{aligned} \tag{18}$$

The equality in (18) holds if there is no loading, i.e., $\delta = 0$; otherwise the strict inequality holds and makes the derivative in (17) or equivalently the derivative in the annuitizing direction negative. Hence, if there is loading on the annuity but not the bond or life insurance then saving will not be annuitized.[6] There is no incentive to shift from bonds to annuities.

Indeed, in the context of this model, we may see the creation of a synthetic annuity. If the investor can go short in life insurance and long in bonds and so move in the direction $v = (0,1,-1)$ [7] then we have

$$\begin{aligned} D_v H &= D_2 H - D_3 H \\ &= -p_b \, M + D_3 u \, D_2 v \, q + \left\{ D_2 u (1-\alpha) + D_3 u \, D_2 v \, \alpha \right\} (1-q) \\ &\quad + p_l \, M - D_3 u \, D_2 v \, q \\ &= \left(p_l - p_b \right) M + \left\{ D_2 u (1-\alpha) + D_3 u \, D_2 v \, \alpha \right\} (1-q) \\ &= \left(p_a + p_l - p_b \right) M \end{aligned} \tag{19}$$

where the last equality follows by (14). Note that

[6] The same conclusion holds given loading on life insurance contracts. See the appendix.
[7] The viatical and life settlement markets suggest that shorting life is possible. There are also other instruments that are substitutes for annuities such as reverse mortgages.

$$\begin{aligned} p_a + p_l - p_b &= \frac{1-q}{(1-\delta)(1+r)} + \frac{q}{1+r} - \frac{1}{1+r} \\ &= p_b\left(\frac{1-q}{(1-\delta)} - (1-q)\right) \\ &\geq 0 \end{aligned} \tag{20}$$

where equality hold if there is no loading and otherwise the strict inequality holds. Hence, there is an incentive to create a synthetic annuity.

There are some other modifications of the model that should be considered. A joint annuity may be introduced into the analysis. This should temper the results. Since longevity risk introduces an aggregate risk[8] for insurers, it seems natural to include the possibility of financial distress for the annuity provider; that financial distress must be considered by the annuitant because it jeopardizes the promise in the annuity contract. Since the life annuity contract cannot be sold by the annuitant, such a contract limits the options available to the buyer; if the buyer anticipates health problems then purchasing annuities may become less desirable. These lines of inquiry are followed in the next sections.

4. Financial Distress

Next, consider the insurer's insolvency risk and its possible impact on the demand for annuities; equivalently, how good is the insurer's promise to provide a cash flow into the indefinite future? This is a particularly important question because the price of securing a reasonable but indefinite cash flow can be quite high. Consider, for example, an immediate annuity that promises $100,000 per year for as long as the individual; such an annuity would cost a 66 year old male approximately $1.2 million *now*.

In order to explore this question in the context of a very simple partial partial equilibrium model, suppose we alter the annuity to make it risky. Suppose the annuity provider promises a one dollar payoff *then* if the individual investor survives and the annuity provider is solvent. Suppose the annuity provider is solvent with probability $(1-p)$. This is quite simple and allows us to leave the structure of the insurer and annuity provider unspecified. This notion can also be introduced into the individual choice problem.

When the individual selects an annuity it is with the knowledge that the payoff *then* is

[8] This aggregate risk refers to the possibility that the mortality rates in the insurers' books of business all change; for example, the cure for a common disease may have an impact on the mortality rates for all or many insureds. This is also loosely referred as longevity risk.

$$= \begin{cases} 0 & q \\ 0 & p(1-q) \\ 1 & (1-p)(1-q) \end{cases}$$

This shows that the annuity pays off if and only if the individual survives and the annuity provider is solvent. This payoff structure will have an impact on the annuity pricing. Rather than paying a fair premium of $(1-q)/(1+r)$ for the annuity the competitive price should converge to $p_a = (1-p)(1-q)/(1+r)$ without loading. This reduces the expense *now* but also reduces the payoff *then* and leaves the individual with no payoff in the event that the individual survives but the annuity provider is insolvent. As an initial step we will continue to suppose that the other financial contracts are safe.

The individual can replace the risky annuity with a safe synthetic annuity. The synthetic annuity has a payoff structure that dominates the risky annuity payoff since the synthetic annuity pays even in the event that the individual survives and the annuity provider is insolvent. Given the same pricing structure for the risky annuity and the synthetic annuity the dominance would be clear. The price of the risk annuity, however, is less than that of the synthetic.

The individual's choice problem must be restructured to introduce the annuity provider's insolvency risk. Consumption *now* is affected to the extent that the price of the annuity changes. The consumption *now* for the investor and significant other are as stated in (8) and (9); the annuity price, however, is adjusted for the insolvency risk as noted here.

Next, consider the altered form of the annuity given insolvency risk. The life annuity moves dollars forward but only if the individual survives and the annuity provider is solvent; the cost *now* of each dollar moved forward with the annuity is p_a. It follows that consumption *then* for the individual is

$$c_{1i} = \begin{cases} c_{1id} & q \\ c_{1ils} & (1-p)(1-q) \\ c_{1ilb} & p(1-q) \end{cases}$$

$$= \begin{cases} 0 & q \\ (1-\alpha)(\lambda_a + \lambda_b) & (1-p)(1-q) \\ (1-\alpha)\lambda_b & p(1-q) \end{cases} \qquad (1)$$

where c_{1id} is consumption *then* in the death event, c_{1ils} is consumption *then* in the living and solvency events, and c_{1ilb} is consumption *then* in the living and insolvency events. For the significant other the consumption *then* is

$$c_{1s} = \begin{cases} c_{1sd} & q \\ c_{1sls} & (1-p)(1-q) \\ c_{1slb} & p(1-q) \end{cases} \tag{2}$$

$$= \begin{cases} \lambda_b + \lambda_l & q \\ \alpha\left(\lambda_a + \lambda_b\right) & (1-p)(1-q) \\ \alpha\lambda_b & p(1-q) \end{cases}$$

where c_{1sd}, c_{1sls} *and* c_{1slb} are similarly defined. It follows that the individual's expected utility is $H(\lambda)$ where

$$H(\lambda) = u\left(c_{0i}, c_{1id}, v(c_{0s}, c_{1sd})\right)q + u\left(c_{0i}, c_{1ils}, v(c_{0s}, c_{1sls})\right)(1-p)(1-q) \tag{3}$$

$$+\ u\left(c_{0i}, c_{1ilb}, v(c_{0s}, c_{1slb})\right)p(1-q)$$

Next consider how the optimum conditions here compare with that of the classic paradigm, i.e., the selfish or egocentric behavior in the last section. The first order conditions are as follows:

$$\begin{aligned} D_1H &= \left\{D_1u\,(1-\alpha)\left[-p_a\right] + D_3u\,D_1v\,\alpha\left[-p_a\right]\right\}q \\ &\quad + \begin{Bmatrix} D_1u\left[\left(1-\alpha\right)\left(-p_a\right)\right] \\ +D_2u\left(1-\alpha\right) + D_3u\,D_1v\left[\alpha\left(-p_a\right)\right] + D_3u\,D_2v\,\alpha \end{Bmatrix}(1-p)\left(1-q\right) \\ &\quad + \left\{D_1u\left[\left(1-\alpha\right)\left(-p_a\right)\right] + D_3u\,D_1v\left[\alpha\left(-p_a\right)\right]\right\}p\left(1-q\right) \\ &= -p_a\,M + \left\{D_2u\left(1-\alpha\right) + D_3u\,D_2v\,\alpha\right\}(1-p)\left(1-q\right) \end{aligned} \tag{4}$$

where the expected marginal utility of consumption now is defined as

$$M \equiv \begin{Bmatrix} \left\{D_1u\,(1-\alpha) + D_3u\,D_1v\,\alpha\right\}q + \left\{D_1u\left(1-\alpha\right) + D_3u\,D_1v\,\alpha\right\}(1-p)\left(1-q\right) \\ +\left\{D_1u\left(1-\alpha\right) + D_3u\,D_1v\,\alpha\right\}p\left(1-q\right) \end{Bmatrix}$$

$$
\begin{aligned}
D_2 H &= \left\{ D_1 u\,(1-\alpha)\left[-p_b\right] + D_3 u\, D_1 v\,\alpha\left[-p_b\right] + D_3 u\, D_2 v \right\} q \\
&\quad + \left\{ \begin{array}{l} D_1 u\left[(1-\alpha)(-p_b)\right] \\ + D_2 u\,(1-\alpha) + D_3 u\, D_1 v\left[\alpha(-p_b)\right] + D_3 u\, D_2 v\,\alpha \end{array} \right\} (1-p)(1-q) \\
&\quad + \left\{ D_1 u\left[(1-\alpha)(-p_b)\right] + D_2 u\,(1-\alpha) + D_3 u\, D_1 v\left[\alpha(-p_b)\right] + D_3 u\, D_2 v\,\alpha \right\} p(1-q) \qquad (5) \\
&= -p_b M + D_3 u\, D_2 v\, q + \left\{ D_2 u\,(1-\alpha) + D_3 u\, D_2 v\,\alpha \right\} (1-p)(1-q) \\
&\quad + \left\{ D_2 u\,(1-\alpha) + D_3 u\, D_2 v\,\alpha \right\} p(1-q)
\end{aligned}
$$

$$
\begin{aligned}
D_3 H &= \left\{ D_1 u\,(1-\alpha)\left[-p_l\right] + D_3 u\, D_1 v\,\alpha\left[-p_l\right] + D_3 u\, D_2 v \right\} q \\
&\quad + \left\{ D_1 u\,(1-\alpha)\left[-p_l\right] + D_3 u\, D_1 v\,\alpha\left[-p_l\right] \right\} (1-p)(1-q) \qquad (6) \\
&\quad + \left\{ D_1 u\,(1-\alpha)\left[-p_l\right] + D_3 u\, D_1 v\,\alpha\left[-p_l\right] \right\} p(1-q) \\
&= -p_l M + D_3 u\, D_2 v\, q
\end{aligned}
$$

As previously noted, moving in the direction $v = (0,1,-1)$ creates a synthetic annuity. Hence, suppose the individual moves in the direction $v = (-1,1,-1)$; this eliminates the annuity subject to insolvency risk and replaces it with the synthetic annuity. If the synthetic annuity dominates the annuity subject to insolvency risk then the derivative in this direction should be positive. Hence consider the derivative of expected utility in the direction $v = (-1,1,-1)$.

$$
\begin{aligned}
D_v H &= -D_1 H + D_2 H - D_3 H \\
&= p_a M - \left\{ D_2 u\left(1-\alpha\right) + D_3 u\, D_2 v\, \alpha \right\}(1-p)\left(1-q\right) \\
&\quad - p_b M + D_3 u\, D_2 v\, q + \left\{ D_2 u\left(1-\alpha\right) + D_3 u\, D_2 v\, \alpha \right\}(1-p)\left(1-q\right) \\
&\quad + \left\{ D_2 u\left(1-\alpha\right) + D_3 u\, D_2 v\, \alpha \right\} p\left(1-q\right) \\
&\quad + p_l M - D_3 u\, D_2 v\, q \\
&= \left(p_a - p_b + p_l \right) M + \left\{ D_2 u\left(1-\alpha\right) + D_3 u\, D_2 v\, \alpha \right\} p\left(1-q\right) \\
&= \left(\frac{\left\{ D_2 u\left(1-\alpha\right) + D_3 u\, D_2 v\, \alpha \right\} p\left(1-q\right)}{M} - \frac{p\left(1-q\right)}{1+r} \right) M
\end{aligned}
\tag{7}
$$

The first term in parentheses on the RHS of (7) is the marginal value or benefit of consumption in the event that the individual survives and the annuity provider does not; the second term in parentheses on the RHS of (7) is the marginal cost of generating a dollar in consumption in the same event. The derivative in the direction $v = \left(-1, 1, -1\right)$ is positive if the marginal benefit exceeds the marginal cost. It should be noted that the marginal utility of consumption in the event that the individual survives and the annuity provider does not is unbounded for the individual and the significant other when there are no bonds in the portfolio, i.e., $D_2 u(c_{0i}, 0, v) = \infty$ and $D_2 v(c_{0s}, 0) = \infty$. It follows that the derivative in the direction that replaces the annuity with the synthetic annuity is positive for a portfolio of the form $\lambda = \left(+, 0, +\right)$. Hence this line of argument shows that the annuity does not dominate the bond. This conclusion is somewhat different from that of (Lopes and Michaelides 2007) who in analyzing rare events such as annuity provider bankruptcy note ". . . these rare events are unlikely candidates to explain the low take-up of voluntary annuities . . ." Although participation rates are too much of a stretch for this analysis, it does show that financial insolvency does play a role in determining annuity demand and that bonds are not dominated even in the absence of altruism.

5. Health Risk Shock

Health risks have been discussed in the literature as a potential explanation of the annuity puzzle, e.g., see (Sinclair and Smetters 2004) and (Davidoff, Brown et al. 2005). Sinclair and Smetters consider a health shock but allow annuity contracts to be liquid or equivalently reversible. Their results demonstrate a diminished annuity demand in the presences of health risk but that diminished demand is due to a reduced life expectancy given the health shock. Indeed even if the results of a genetic test provided cause for a reduced life expectancy then individual demand for annuities would decrease without the shock of a large medical expenditure. Unlike Sinclair and Smetters, the annuity here is supposed to be illiquid. Also unlike Sinclair and Smetters, life expectancy is not assumed to be affected by the health shock. It is the illiquidity of the annuity alone that drives the result. Indeed in this setting the demand for annuities should disappear since the synthetic equivalent dominates.

The health risk in a model with a longer time horizon than that considered here may occur *then* or *later*. The feature of the annuity of interest here is that the annuity requires a large investment *now* and generates a smaller stream of income into the indefinite future. Because it is illiquid it diminishes the individual's ability to generate a large amount of cash for a medical expenditure at any date in that indefinite future. To capture the essence of this story in a two date model, suppose that the health risk exists *now* rather than *then* or *later*. The illiquidity of an annuity works much the same way in this context as it would in a multi-period setting. With the health risk *now*, suppose the wealth *now* is w but the individual faces a medical expense of L dollars with probability θ and zero dollars otherwise. This makes initial wealth random, i.e.,

$$W = \begin{cases} w & 1-\theta \\ w-L & \theta \end{cases}$$

If the individual must make the portfolio decision $\lambda = (\lambda_a, \lambda_b, \lambda_l)$ before the health risk is resolved then the problem is manifest. In this case there is a decision sequence. First a portfolio is selected, second the health risk is resolved and third the portfolio is rebalanced if the medical expense occurs.

At least part of the explanation is motivated by the notion that if the individual faces the risk of a large expenditure due to the treatment of a medical condition then a prudent individual would allocate funds so that the medical expense could be covered. Such an allocation of funds would compete with the demand for annuities. The annuity and bond move money forward and the annuity allows potentially more to be forwarded for the same dollar investment. If the health risk exists *now* rather than *then*, then purchasing an annuity would compete with having cash or cash equivalent available. If the annuity cannot be reversed or sold then it is not liquid. A bond, however, is liquid; it can be purchased and then sold or purchased and then borrowed against. Hence, the individual retains liquidity by purchasing the bond rather than the annuity; if the annuity was desired then the synthetic annuity with its long position in the bond would dominate the annuity.

6. Concluding Remarks

The literature contains some important insights that suggest plausible reasons for a limited life annuity demand. The models constructed in the literature have been used to integrate the bequest motive, adverse selection, aggregate mortality risk and incomplete markets but no consensus has emerged that explains the annuity puzzle. An altered paradigm is clearly necessary. A simple theoretical construct is presented here that incorporates altruism. This construct provides a foundation for life insurance demand that is missing in the classic economic model and this construct, like the bequest motive, shows that the marginal rate of substitution due to altruism can be sufficient to eliminate the annuity demand. The analysis is extended to show that an annuity provider's insolvency risk and an investor's health risk also play a role in explaining the puzzle.

7. Appendix

Suppose there is loading δ on the annuity and γ is the loading on the life contract. Then moving in the annuitizing direction $v=(1,-1,0)$ yields

$$\begin{aligned} D_v H &= \left(p_b - p_a - p_l\right)M \\ &= \left\{ p_b\left(1-\frac{1-q}{1-\delta}\right) - \frac{q}{(1-\gamma)(1+r)} \right\} M \\ &= \left\{ p_b\left[\frac{q-\delta}{1-\delta} - \frac{q}{1-\gamma}\right]\right\} M \\ &= \left\{ p_b\left[-\frac{\delta}{(1-\delta)} + q\left(\frac{1}{1-\delta} - \frac{1}{1-\gamma}\right)\right]\right\} M \\ &= \left\{ p_b\left[-\frac{\delta}{(1-\delta)} + q\frac{\delta-\gamma}{(1-\delta)(1-\gamma)}\right]\right\} M \\ &< 0 \end{aligned} \tag{8}$$

If $\gamma \geq \delta$ then (8) clearly holds. If $\gamma < \delta$ then

$$-\frac{\delta}{(1-\delta)} + q\frac{\delta-\gamma}{(1-\delta)(1-\gamma)} < 0 \Leftrightarrow q\frac{\delta-\gamma}{(1-\delta)(1-\gamma)} < \frac{\delta}{(1-\delta)} \Leftrightarrow q < \frac{\delta(1-\gamma)}{\delta-\gamma} \tag{9}$$

and the last inequality clearly holds since the right hand side is greater than one. Hence, (8) holds for all γ.

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