Duration Notes[1]

Macaulay defined the duration of an asset in 1938.[2] Let the stream of payments constituting the asset be $a_0, a_1, \ldots, a_T$ and let $\delta = 1/(1 + r)$ denote the discount factor. Then Macaulay's definition of the duration of the asset is[3]

$$D = \frac{\sum_{t=0}^{T} t\delta^t a_t}{\sum_{t=0}^{T} \delta^t a_t} = \frac{\delta a_1 + 2\delta^2 a_2 + \ldots + T\delta^T a_T}{a_0 + \delta a_1 + \delta^2 a_2 + \ldots + \delta^T a_T}$$

To motivate this measure, observe that the duration may also be expressed as

$$D = \sum_{t=0}^{T} t \frac{\delta^t a_t}{\sum_{t=0}^{T} \delta^t a_t} = = \sum_{t=0}^{T} t \frac{\delta^t a_t}{A}$$

where A is the present value of the asset, i.e.,

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[1] Weil seems to attribute the origination of the duration concept to Macaulay. The notion of an average period of production goes back farther. For example, Bohm-Bawerk defined an average period of production which is identical to Hicks' average period and so to Macaulay's. Bohm-Bawerk is attributed with the introduction of the concept of the average period of production and by this he meant ". . . the average time distance between original inputs and the final product" (See C. C. von Weizsacker, *Steady State Capital Theory*, Springer-Verlag 1971, p. 33). Bohm-Bawerk assumed that output was an increasing function of time, i.e., q(T) where q' > 0. In a simple point input - point output model, it is clear that the T selected is the average period of production. Note that using Macaulay's measure of duration, we obtain

$$D = \frac{T\,q(T)\,e^{-rT}}{q(T)\,e^{-rT}} = T$$

Hence, we may trace the notion of average period of production or equivalently duration back at least as far as Bohm-Bawerk's *Positive Theory of Capital*.

[2] Frederick R. Macaulay, Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the United States since 1856 (New York: Columbia University Press for the National Bureau of Economic Research, 1938).

[3] I am not sure that Macaulay allowed for the date t = 0 but I am including it so that the comparison with Hicks' elasticity is precise.

$$A = \sum_{t=0}^{T} \delta^t a_t$$

Since

$$\sum_{t=0}^{T} \frac{\delta^t a_t}{A} = 1$$

the duration measure can be understood as the mean period of the asset. In 1939, Hicks noted that the average period of an asset, i.e.,

$$\frac{\delta a_1 + 2\delta^2 a_2 + \ldots + T\delta^T a_T}{a_0 + \delta a_1 + \delta^2 a_2 + \ldots + \delta^T a_T}$$

is the elasticity of the asset value with respect to the discount factor. Hick noted that this elasticity " . . . is the average length of time for which the various payments are are deferred from the present, when the times of deferment are weighted by the discounted values of the payments."[4] To see this, note that the elasticity of the asset value with respect to the discount factor is the percentage change in asset value per percentage change in the discount factor, i.e.,

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[4]See J. R. Hicks, *Value and Capital*, (Oxford: Clarendon Press, 1939), p. 186.

$$\varepsilon_\delta \equiv \frac{\frac{dA}{A}}{\frac{d\delta}{\delta}} = \frac{dA}{d\delta}\frac{\delta}{A} = \frac{\delta}{A}\sum_{t=0}^{T} t\,\delta^{t-1}\,a_t = \frac{\sum_{t=0}^{T} t\delta^t a_t}{\sum_{t=0}^{T}\delta^t a_t} = D$$

Hence Macaulay's duration and Hicks' average period of an asset are equivalent concepts. Alternatively, the duration may be expressed using the elasticity of the asset value with respect to the rate of interest. Direct calculation shows

$$\varepsilon_r = \frac{\frac{dA}{A}}{\frac{dr}{r}} = \frac{dA}{dr}\frac{r}{A} = \frac{r}{A}\sum_{t=0}^{T} t\,\delta^{t-1}\frac{d\delta}{dr}a_t = -\delta\, r\sum_{t=0}^{T} t\,\frac{\delta^t a_t}{A} = -\,\delta\, r\, D,$$

or equivalently,

$$D = -\frac{\varepsilon_r}{\delta\, r}$$

where ε_r is the elasticity of the asset value with respect to the interest rate r. Note that the rate of change in the value of the asset with respect to the interest rate is

$$\frac{dA}{dr} = \frac{A}{r}\,\varepsilon_r = \frac{A}{r}\left(-\,\delta\, r\, D\right) = -\,\delta\, D\, A \qquad (1)$$

The duration of a perpetuity with a one dollar payoff in each period beginning with t = 1 is

$$D = \varepsilon_\delta = \frac{\delta + 2\delta^2 + \ldots + T\delta^T + \ldots}{\delta + \delta^2 + \ldots + \delta^T + \ldots} = \frac{\dfrac{\delta}{(1-\delta)^2}}{\dfrac{\delta}{(1-\delta)}} = \frac{1}{1-\delta} = \frac{1+r}{r}$$

and so an interest rate of 5% yields a duration of 21. It may be noted that this duration is also the minimum number of periods required to acquire the present value of the perpetuity in cash, i.e., since the present value of the perpetuity is 20. In this case, the rate of change in the present value of the perpetuity is

$$\frac{dA}{dr} = -\ \delta DA = -\ \delta \frac{1}{1-\delta} \frac{\delta}{1-\delta} = -\left(\frac{\delta}{1-\delta}\right)^2 = -\ A^2$$

and the interest elasticity of the perpetuity is

$$\varepsilon_r = -\ \delta\ r\ D = -\frac{1}{1+r}\ r\ \frac{1+r}{r} = -\ 1$$

Equivalently,

$$\frac{dA}{A} = -\ \frac{dr}{r}$$

and so a one percent increase in the interest rate yields a one percent decrease in the value of the perpetuity.

Weil says that the earliest paper he is aware of on immunization is Tjalling C. Koopmans' work, *The Risk of Interest Fluctuations in Life Insurance Companies*.[5] However,

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[5]This was written while Koopmans worked for Penn Mutual Life Insurance Company and has not been published. See R. L. Weil, "Macaulay's Duration: An Appreciation," *Journal of Business*, (October 1973).

Weil also notes that in 1952, Redington defined the mean term of an asset stream and a liability stream, and showed that the payoff of an insurance company is immune to small changes in the interest rate if the mean term of the asset equals that of the liability.[6] Redington's definition of mean term is

$$\sum_{t=0}^{T} t\,\delta^{t}\,a_t$$

Letting $l_0, l_1, \ldots, l_{T_l}$ denote the sequence of liabilities and $a_0, a_1, \ldots, a_{T_a}$ the sequence of asset payoffs, it follows that Redington showed that the insurance company is immunized if

$$\sum_{t=0}^{T_a} t\,\delta^{t}\,a_t = \sum_{t=0}^{T_l} t\,\delta^{t}\,l_t \qquad (2)$$

Now let L denote the present value of the liability stream, i.e.,

$$L = \sum_{t=0}^{T_l} \delta^{t}\,l_t$$

and let D_L and D_A denote the duration of the liability and asset, respectively. Then, stating Redington's analysis in terms of duration, note that (2) is equivalent to the following equation (3),

$$D_A\,A = D_L\,L. \qquad (3)$$

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[6]See F. M. Redington, "Review of the Principles of Life-Office Valuations," *Journal of the Institute of Actuaries*, (1952).

From (1) it is also clear that (3) may be equivalently expressed as

$$-\frac{dA}{dr}\frac{1}{\delta} = -\frac{dL}{dr}\frac{1}{\delta} \Leftrightarrow \frac{dA}{dr} = \frac{dL}{dr} \qquad (4)$$

Hence, it is quite clear that if Redington's condition (2) is satisfied then the firm is immunized. Alternatively, let the value of the institution or corporation be expressed as V, where V = A - L. Then, using (1), it follows that

$$\frac{dV}{dr} = \frac{dA}{dr} - \frac{dL}{dr} = \left(-\delta D_A A\right) - \left(-\delta D_L L\right) = \delta \left(D_L L - D_A A\right)$$

Therefore,

$$\frac{dV}{dr} \lesseqgtr 0 \text{ as} \left(D_L L - D_A A\right) \lesseqgtr 0$$

Although Samuelson was apparently unaware of Macaulay's work, this is essentially what he showed in 1945.[7] Samuelson showed that if the weighted average time period of the cash outflow exceeds that of the cash inflow then the institution will profit when interest rates rise. Samuelson's weighted average time period is equivalent to D_L L and D_A A, for outflows and inflows, respectively.

In 1957, Durand argued that the only financial assets with long durations were growth stocks. Then he argued that institutions with long duration liabilities need to hold growth

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[7]See Paul A. Samuelson, "The Effects of interest Rate Increases on the Banking System," *American Economic Review* 35 (March 1945), 16-27. This article is online at the following url: <http://index2.umdl.umich.edu/cgi-bin/jstor/viewitem.stable/00028282/di950313/95p02137/0?frame=frame&userID=NoUserID&dpi=3&config=jstor>

stocks to reduce the risk of financial loss from interest rate fluctuations.[8] It is apparent that the institution can immunize by using growth stock. The duration measure for the growth stock is

$$D = \frac{(1+r)\ a_1}{r-g}$$

This duration is larger than that for a no growth stock with a level payment stream a_1, i.e.,

$$\frac{(1+r)\ a_1}{r-g} > \frac{(1+r)\ a_1}{r}$$

In the absence of immunization, it is also apparent that the value of the institution will only fall if the interest rate falls.

Duration Mathematics

Duration of an Annuity

First consider the duration of an annuity. Let $a_t = a$ for $t = 1, \ldots, T$ denote the constant cashflow and let A denote the value of the annuity. Then direct calculation shows that the duration of the annuity is D_A where

$$D_A = \sum_{t=1}^{T} t \frac{\dfrac{a}{(1+r)^t}}{A} = \frac{1+r}{r} - \frac{1}{r\,(1+r)^T} \frac{T\ a}{A}$$

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[8]See David Durand, "Growth Stocks and the Petersburg Paradox," *Journal of Finance* 12, (September 1957), 348-63.

Duration of a Bond

Let a denote the interest per period and f denote the face value at maturity T. Similarly let A denote the value of the interest payments and F denote the present value of the face value. Finally, let B denote the bond value. Then the bond duration D_B is

$$D_B = \sum_{t=1}^{T} t \frac{\delta_t a}{B} + T \frac{\delta_T f}{B}$$

$$= \left(\sum_{t=1}^{T} t \frac{\delta_t a}{A} \right) \frac{A}{B} + T \frac{\delta_T f}{F} \frac{F}{B}$$

$$= D_A \frac{A}{B} + D_F \frac{F}{B}$$

Hence, the bond duration is the weighted average of the duration of the annuity representing the interest payments and the duration of the principal payment.

Duration of a Sum

Let A represent the sum of asset values $A_1, A_2, \ldots, A_n$. Similarly, let a_t represent the cashflow of the sum of the assets at date t, i.e.,

$$a_t = \sum_{j=1}^{n} a_{jt}$$

The duration of the sum is a weighted average of the asset durations since

$$D_A = \sum_{t=1}^{T} t \frac{\delta_t a_t}{A}$$

$$= \sum_{j=1}^{n} \sum_{t=1}^{T} t \frac{\delta_t a_{jt}}{A}$$

$$= \sum_{j=1}^{n} \frac{A_j}{A} \left(\sum_{t=1}^{T} t \frac{\delta_t a_{jt}}{A_j} \right)$$

$$= \sum_{j=1}^{n} \frac{A_j}{A} D_{A_j}$$

Similarly, the duration of a difference is the weighted sum of the difference of duration measures.

Duration of a Portfolio

Let $A = \alpha_1 A_1 + \ldots + \alpha_n A_n$ represent the value of the portfolio $\alpha = (\alpha_1, \ldots, \alpha_n)$ of assets. Then the duration of the portfolio is a weighted sum of the asset durations.

$$D_A = \sum_{j=1}^{n} \sum_{t=1}^{T} t \frac{\delta_t \alpha_j a_{jt}}{A}$$

$$= \sum_{j=1}^{n} \alpha_j \frac{A_j}{A} \left(\sum_{t=1}^{T} t \frac{\delta_t a_{jt}}{A_j} \right)$$

$$= \sum_{j=1}^{n} \alpha_j \frac{A_j}{A} D_{A_j}$$